TROPICAL TYPES AND
ASSOCIATED CELLULAR RESOLUTIONS

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ABSTRACT. An arrangement of finitely many tropical hyperplanes in the tropical torus $\mathbb{T}^{d-1}$ leads to a notion of 'type' data for points in $\mathbb{T}^{d-1}$, with the underlying unlabeled arrangement giving rise to 'coarse type'. It is shown that the decomposition of $\mathbb{T}^{d-1}$ induced by types gives rise to minimal cocellular resolutions of certain associated monomial ideals. Via the Cayley trick from geometric combinatorics this also yields cellular resolutions supported on mixed subdivisions of dilated simplices, extending previously known constructions. Moreover, the methods developed lead to an algebraic algorithm for computing the facial structure of arbitrary tropical complexes from point data.

1. INTRODUCTION

The study of convexity over the tropical semiring has been an area of active research in recent years. Fundamental properties of tropical convexity, in particular from a combinatorial perspective, were established by Develin and Sturmfels in [7]. There the notion of a tropical polytope was defined as the tropical convex hull of a finite set of points in the tropical torus $\mathbb{T}^{d-1}$. Fixing the set of generating points yields a decomposition of the tropical polytope called the *tropical complex*, and in [7] it was shown that the collection of such complexes are in bijection with the regular subdivisions of a product of simplices. The origin of tropical convexity can be traced back to the study of 'max-plus linear algebra'; see [18, 5] and the references therein for a proper account.

A tropical complex can be realized as the subcomplex of bounded cells of the polyhedral complex arising from an arrangement of tropical hyperplanes, and it is this perspective that we adopt in this paper. We study combinatorial properties of arrangements of tropical hyperplanes, and in particular their relation to algebraic properties of associated monomial ideals. A tropical hyperplane in $\mathbb{T}^{d-1}$ is defined as the locus of 'tropical vanishing' of a linear form, and can be regarded as a fan polar to a $(d-1)$-dimensional simplex. In this way each tropical hyperplane divides the ambient space $\mathbb{T}^{d-1}$ into d *sectors*. Given an arrangement $\mathcal{A}$ of tropical hyperplanes and a point $p \in \mathbb{T}^{d-1}$, one can record the position of p with respect to each sector of each hyperplane. This tropical analog of the covector data of an oriented matroid is called the *type* data, and the combinatorial approach to tropical convexity taken in [7] is based on this concept. Here we consider a coarsening of the type data (which we call *coarse type*) arising from an arrangement $\mathcal{A}$ of tropical hyperplanes, amounting to neglecting the labels on the individual hyperplanes.

The connection between tropical polytopes/complexes and resolutions of monomial ideals was first exploited by Block and Yu in [4]. There the authors associate a monomial ideal to

a tropical polytope with generators in general position, and use algebraic properties of its minimal resolution to determine the facial structure of the bounded subcomplex. Further progress along these lines was made in [8]. The primary tool employed in this context is that of a *cellular resolution* of a monomial ideal I, where the i-th syzygies of I are encoded by the i-dimensional faces of a polyhedral complex (see Section 3). The ideals from [4] are squarefree monomials ideals generated by the *cotype* data, i.e. the complements of the tropical covectors, arising from the associated arrangement of hyperplanes, and hence can be seen as a tropical analog of the *(oriented) matroid ideals* studied by Novik, Postnikov and Sturmfels in [22].

In this paper we study the polyhedral complex $\mathcal{C}_{\mathcal{A}}$ (and its bounded subcomplex $\mathcal{B}_{\mathcal{A}}$) induced by the type data of an arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{T}^{d-1}$. Both complexes are naturally labeled by fine and coarse type and cotype data, and we show how the resulting labeled complexes support minimal (co)cellular resolutions of associated monomial ideals. We pay special attention to labels given by coarse type. For instance, we show that $\mathcal{C}_{\mathcal{A}}$ supports a minimal *cocellular* resolution of the ideal $I_{\mathbf{t}(\mathcal{A})}$ generated by monomials corresponding to the set of all coarse types. The proof involves a consideration of the topology of certain subsets of $\mathcal{C}_{\mathcal{A}}$ as well as the combinatorial properties of the coarse type labelings. When the arrangement $\mathcal{A}$ is *sufficiently generic* we show that the resulting ideal is always given by $\langle x_1, \ldots, x_d \rangle^n$, the n-th power of the maximal homogeneous ideal; in general, $I_{\mathbf{t}(\mathcal{A})}$ is some Artinian subideal. Our results in this area are all independent of the characteristic of the coefficient field.

Via the connection to products of simplices and the Cayley trick we interpret these results in the context of mixed subdivisions of dilated simplices. In particular we obtain a minimal *cellular* resolution of $I_{\mathbf{t}(\mathcal{A})}$ supported on a subcomplex of the dilated simplex $n\Delta_{d-1}$. One other direct consequence is that *any* regular fine mixed subdivision of $n\Delta_{d-1}$ supports a minimal resolution of $\langle x_1, \ldots, x_d \rangle^n$. This extends a result of Sinefakopoulos from [29] where a particular subdivision is considered (although much less explicitly), and also complements a construction of Engström and the first author from [9] where such complexes are applied to resolutions of hypergraph edge ideals. The duality between tropical complexes and mixed subdivisions of dilated simplices was established in [7], and we show how this extends to the algebraic level in terms of Alexander duality of our resolutions of the coarse type and cotype ideals.

Finally, we show how these algebraic results lead to observations regarding the combinatorics of tropical polytopes/complexes and mixed subdivisions of dilated simplices. We obtain a formula for the f-vector of the bounded subcomplex of an arbitrary tropical hyperplane arrangement in terms of the Betti numbers of the associated coarse type ideal. The uniqueness of minimal resolutions also implies that for any sufficiently generic arrangement $\mathcal{A}$, the multiset of coarse types is independent of the arrangement. Furthermore, we present an algorithm for determining the incidence face structure of a tropical complex from the coordinates of the generic set of vertices, utilizing the fact that such a complex supports a minimal resolution of the square-free monomial cotype ideal. This approach was first introduced by Block and Yu in [4] for the case of sufficiently generic arrangements, and we extend the algorithm to the general case.

The rest of the paper is organized as follows. In Section 2 we review the basic notions of tropical convexity including tropical hyperplanes and type data, and discuss the polytopal complex that arises from an arrangement of tropical hyperplanes. We introduce the notion of *coarse type* and establish some of the basic properties that will be used later in the paper. In Section 3 we introduce the monomial ideals that arise from an arrangement of hyperplanes and show that the polytopal complexes labeled by fine and coarse (co)type support cocellular

(and cellular) resolutions. In Section 4 we recall the construction of the Cayley trick and use this to interpret our results in terms of mixed subdivisions of dilated simplices. In Section 5 we discuss some examples of our construction, in particular the case of the (generic) *staircase triangulation* (recovering a result of [29]) as well as a family of non-generic arrangements corresponding to tropical *hypersimplices*. In Section 6 we show our results give rise to certain consequences for the combinatorics (e.g., the f-vector) of the bounded subcomplexes of tropical hyperplane arrangements, and also describe an algorithm for determining the entire face poset from the coordinates of the arrangement. This strengthens a result from [4]. Finally, we end in Section 7 with some concluding remarks and open questions.

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2. TROPICAL CONVEXITY AND COARSE TYPES

In order to fix our notation we begin in this section with a brief review of the foundations of tropical convexity as layed out by Develin and Sturmfels in [7]. We then define 'coarse types' and establish some combinatorial and topological results regarding the type decomposition of the tropical torus induced by a finite set of points. While some of these observations may be worthwhile in their own right, their main interest for us will be their applications to subsequent constructions of (co)cellular resolutions.

2.1. **Tropical convexity and tropical hyperplane arrangements.** Tropical convexity
is concerned with linear algebra over the *tropical semi-ring* $(\mathbb{R}, \oplus, \odot)$, where

$$x \oplus y \; := \; \min(x, y) \quad \text{and} \quad x \odot y \; := \; x + y \,.$$

We will sometimes replace the operation min with max, and although the two resulting semi-rings are isomorphic via

$$-\max(x, y) \; = \; \min(-x, -y) \,,$$

it will be useful for us to consider both structures on the set $\mathbb{R}$ simultaneously. To avoid confusion we will therefore use the terms *min-tropical semi-ring* and *max-tropical semi-ring*, respectively. Componentwise tropical addition and tropical scalar multiplication turn $\mathbb{R}^d$ into a semi-module. The *tropical torus* $\mathbb{T}^{d-1}$ is the quotient of Euclidean space $\mathbb{R}^d$ by the linear subspace $\mathbb{R}\mathbb{1}$, where $\mathbb{1} \in \mathbb{R}^d$ is the all-ones vector. By interpreting this quotient in the category of topological spaces, $\mathbb{T}^{d-1}$ inherits a natural topology which is homeomorphic to the usual topology on $\mathbb{R}^{d-1}$. A set $S \subset \mathbb{T}^{d-1}$ is *tropically convex* if it contains $(\lambda \odot x) \oplus (\mu \odot y)$ for all $x, y \in S$ and $\lambda, \mu \in \mathbb{R}$. For an arbitrary set $S \subset \mathbb{T}^{d-1}$ the *tropical convex hull* tconv(S) is defined as the smallest tropically convex set containing S. If the set S is finite, then tconv(S) is called a *tropical polytope*. In this paper, tropical convexity and related notions will be studied with respect to both min and max, and hence we will also talk about *max-tropically convex* sets and the like.

   The *tropical hyperplane* with *apex* $-a \in \mathbb{T}^{d-1}$ is the set

$$H(-a) \; := \; \left\{ p \in \mathbb{T}^{d-1} \, : \, (a_1 \odot p_1) \oplus (a_2 \odot p_2) \oplus \cdots \oplus (a_d \odot p_d) \text{ is attained at least twice} \right\} \,.$$

That is, a tropical hyperplane is the tropical vanishing locus of a polynomial homogeneous of degree 1 with real coefficients. If we want to explicitly distinguish between the min- and max-versions we write $H^{\min}(-a)$ and $H^{\max}(-a)$, respectively. Any two min-tropical (resp. max-tropical) hyperplanes are related by an ordinary translation, and hence a tropical hyperplane is completely determined by its apex. Furthermore, the min-tropical hyperplane with apex 0 is a mirror image of the max-tropical hyperplane with apex 0 under the map

$x \mapsto -x$. The complement of any tropical hyperplane in $\mathbb{T}^{d-1}$ consists of precisely d connected components, its *open sectors*. Each open sector is convex, in both the tropical and ordinary sense. The *k-th (closed) sector* of the max-tropical hyperplane with apex a is the set

$$S_k^{\max}(a) \ := \ \left\{ p \in \mathbb{T}^{d-1} \ : \ a_k - p_k \leq a_i - p_i \text{ for all } i \in [d] \right\} .$$

Similarly we have

$$S_k^{\min}(a) \ := \ \left\{ p \in \mathbb{T}^{d-1} \ : \ a_k - p_k \geq a_i - p_i \text{ for all } i \in [d] \right\}$$

for the min-version. Notice that $x \in S_k^{\max}(y)$ if and only if $y \in S_k^{\min}(x)$. Each closed sector is the topological closure of an open sector. Again the closed sectors are tropically and ordinarily convex. A sequence of points $V = (v_1, v_2, \ldots, v_n)$ in $\mathbb{T}^{d-1}$ gives rise to the arrangement

$$\mathcal{A}(V) \ := \ \left(H^{\max}(v_1), H^{\max}(v_2), \ldots, H^{\max}(v_n) \right)$$

of n labeled max-tropical hyperplanes. The position of points in $\mathbb{T}^{d-1}$ relative to each hyperplane in the arrangement furnishes combinatorial data and leads to the following definition.

Definition 2.1 (Fine type)**.** Let $\mathcal{A} = \mathcal{A}(V)$ be the arrangement of max-tropical hyperplanes given by $V = (v_1, v_2, \ldots, v_n)$ in $\mathbb{T}^{d-1}$. The *fine type* (or sometimes simply *type*) of a point $p \in \mathbb{T}^{d-1}$ with respect to $\mathcal{A}$ is the table $T_{\mathcal{A}}(p) \in \{0, 1\}^{n \times d}$ with

$$T_{\mathcal{A}}(p)_{ik} = 1 \quad \text{if and only if} \quad p \in S_k^{\max}(v_i)$$

for $i \in [n]$ and $k \in [d]$. The *fine cotype* $\overline{T}_{\mathcal{A}}(p) \in \{0, 1\}^{n \times d}$ is defined as the complementary matrix, that is,

$$\overline{T}_{\mathcal{A}}(p)_{ik} = 1 \quad \text{if and only if} \quad T_{\mathcal{A}}(p)_{ik} = 0 .$$

Let us now fix a sequence of points V and the corresponding arrangement $\mathcal{A} = \mathcal{A}(V)$ in $\mathbb{T}^{d-1}$. We write $T(p)$ instead of $T_{\mathcal{A}}(p)$ when no confusion arises. For a fixed type $T = T(p)$, the set

$$C_T^{\circ} \ := \ \left\{ q \in \mathbb{T}^{d-1} \ : \ T(q) = T \right\}$$

of points in $\mathbb{T}^{d-1}$ with that type is a relatively open subset of $\mathbb{T}^{d-1} = \mathbb{R}^{d-1}$; called the *relatively open cell* of type T. The set C_T° as well as its topological closure C_T are tropically and ordinarily convex. The set of all relatively open cells $\mathcal{C}^{\circ} = \mathcal{C}^{\circ}(V)$ partitions the tropical torus $\mathbb{T}^{d-1}$. Moreover, C_T, the *closed cell of type T*, is the intersection of finitely many closed sectors. Since each closed sector is the intersection of finitely many polyhedral cones, this implies that C_T, provided it is bounded, is a polytope in both the classical and the tropical sense; in [14] these were called *polytropes*. The collection of all closed cells yields a polyhedral subdivision $\mathcal{C} = \mathcal{C}(V)$ of $\mathbb{T}^{d-1}$. The collection of types with the componentwise order is anti-isomorphic to the face lattice of $\mathcal{C}(V)$, that is, $T_{\mathcal{A}}(D) \leq T_{\mathcal{A}}(C)$ whenever $C \subseteq D$ are closed cells of $\mathcal{C}(V)$. The cells which are bounded form the *bounded subcomplex* $\mathcal{B} = \mathcal{B}(V)$. Following Ardila and Develin [1], a fine type should be thought of as the tropical equivalent of a covector in the setting of tropical oriented matroids, with the cells of maximal dimension playing the role of the topes. The following result highlights the relation of min-tropical polytopes and max-tropical hyperplane arrangements.

Theorem 2.2 ([7, Thm. 15 and Prop. 16])**.** *The* min-*tropical polytope* $\mathrm{tconv}(V)$ *is the union of cells in the bounded subcomplex* $\mathcal{B}(V)$ *of the cell decomposition of* $\mathbb{T}^{d-1}$ *induced by the* max-*tropical hyperplane arrangement* $\mathcal{A}(V)$.

The polytopal complex $\mathcal{B}(V)$ induced by types is a subdivision of the tropical polytope tconv(P) called the *tropical complex* generated by V. The points V are in tropically *general position* if the combinatorial type of $\mathcal{C}(V)$ (or equivalently $\mathcal{B}(V)$) is invariant under small perturbations. In Section 4, we will elaborate on the important connection between tropical hyperplane arrangements and mixed subdivisions of dilated simplices. In light of this connection, the points V are in general position if the associated mixed subdivision is *fine*.

At this point we introduce our running example, borrowed from [4, Ex. 10].

Example 2.3. The points $v_1 = (0,3,6)$, $v_2 = (0,5,2)$, $v_3 = (0,0,1)$, and $v_4 = (1,5,0)$ give rise to the max-tropical hyperplane arrangement shown in Figure 1. It decomposes the tropical torus $\mathbb{T}^2$ into 15 two-dimensional cells, three of which are bounded. Note that there is precisely one bounded cell of dimension one (incident with the 0-cell v_1) which is maximal with respect to inclusion. This shows that the polytopal complex $\mathcal{B}(V)$ need not be pure. Moreover, one can check that the dual regular subdivision of $\Delta_3 \times \Delta_2$ is a triangulation, and hence these four points are sufficiently generic.

2.2. **Coarse types.** As we have seen, the fine type records the position of a point relative to a labeled tropical hyperplane arrangement. Neglecting the labels on the hyperplanes leads to the following coarsening of the type information.

Definition 2.4 (Coarse type). Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of n max-tropical hyperplanes in $\mathbb{T}^{d-1}$. The *coarse type* of a point $p \in \mathbb{T}^{d-1}$ with respect to $\mathcal{A}$ is given by $\mathbf{t}_{\mathcal{A}}(p) = (t_1, t_2, \ldots, t_d) \in \mathbb{N}^d$ with

$$t_k = \sum_{i=1}^{n} T_{\mathcal{A}}(p)_{ik}$$

for $k \in [d]$.

The coarse type entry t_k records for how many hyperplanes in $\mathcal{A}$ the point p lies in the k-th closed sector. Again we will write $\mathbf{t}(p)$ when no confusion can arise. By definition $\mathbf{t}$ is constant when restricted to a relatively open cell C_T°. The induced map $\mathcal{C}^\circ \to \mathbb{N}^d$ is also denoted by $\mathbf{t}$.

Objects in tropical geometry can be lifted to objects in classical algebraic geometry by considering fields of formal Puiseux series and their valuations. Following this path leads to another interpretation of the coarse types: For $v_i \in \mathbb{T}^{d-1}$ the max-tropical hyperplane $H^{\max}(v_i)$ is the *tropicalization* $\mathrm{trop}^{\max}(h_i)$ of a hyperplane given as the vanishing of a homogeneous linear form $h_i \in \mathbb{C}\{\{z\}\}[x_1, x_2, \ldots, x_d]$ defined over the field of Puiseux series. Let

$$(1) \qquad\qquad h = h_1 \cdot h_2 \cdots h_n$$

be the product of these n linear forms, so that h is a homogeneous polynomial of degree n in d indeterminates. Notice that there is no canonical choice for the polynomials h_i and hence not for h. In the computation below we choose h_i to be $z^{-v_{i1}}x_1 + z^{-v_{i2}}x_2 + \cdots + z^{-v_{id}}x_d$.

Remark 2.5. Note that the classical Puiseux series have rational exponents, and therefore the valuation map takes rational values only. For this reason the tropical hypersurface of a tropical polynomial is often defined as the topological closure of the vanishing locus of a tropical polynomial; e.g., see Einsiedler, Kapranov, and Lind [11]. By extending $\mathbb{C}\{\{z\}\}$ to a field of *generalized Puiseux series* one can directly deal with real exponents, as the corresponding valuation map is onto the reals; for a construction see Markwig [19]. In
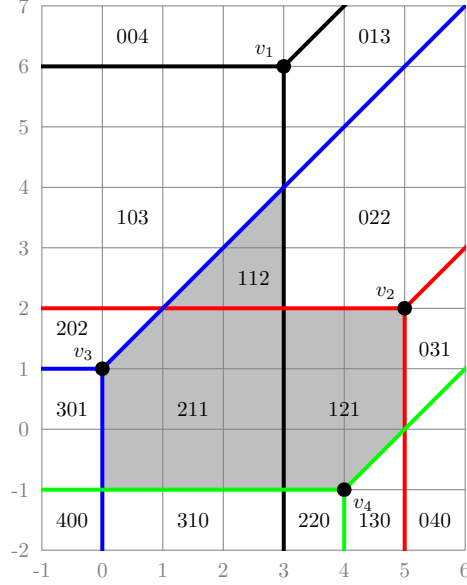
FIGURE 1. Coarsely labeled type decomposition of $\mathbb{T}^2$ induced by four max-tropical lines. Bounded cells are shaded.

the sequel we will make use of such a field of generalized Puiseux series, and we denote it by $\mathbb{C}\{\{z\}\}^{\text{gen}}$.

Proposition 2.6. *The tropical hypersurface defined by* $\text{trop}^{\max}(h)$ *is the union of the* max-*tropical hyperplanes in* $\mathcal{A}$.

The tropical hypersurface defined by $\text{trop}^{\max}(h)$ is the orthogonal projection of the co-dimension-2-skeleton of an unbounded ordinary convex polyhedron $P_{\mathcal{A}}$ in $\mathbb{R}^d$ whose facets correspond to monomials of the polynomial h; see [24, Thm. 3.3].

Proposition 2.7. *Let* $p \in \mathbb{T}^{d-1} \setminus \mathcal{A}$ *be a generic point. Then its coarse type* $\mathbf{t}_{\mathcal{A}}(p)$ *is the exponent of the monomial in* h *which defines the unique facet of* $P_{\mathcal{A}}$ *above* p.

Proof. Since p is generic, it is contained in a unique sector $S^{\max}_{\tau_i}(v_i)$ with respect to the hyperplane with apex v_i. Hence, p satisfies the strict inequalities

(2) $$v_{i,\tau_i} - p_{\tau_i} \ < \ v_{i,j} - p_j$$

for all $i \in [n]$ and $j \in [d] \setminus \tau_i$. Equivalently,

$$\sum_{i=1}^{n} p_{\tau_i} - v_{i,\tau_i}$$

is the evaluation of the tropical polynomial $\text{trop}^{\max}(h)$ and the corresponding monomial in h at which the unique maximum is attained is $x_{\tau_1} x_{\tau_2} \cdots x_{\tau_n}$ with coefficient $z^{-v_{1,\tau_1} - v_{2,\tau_2} - \cdots - v_{n,\tau_n}}$.
□

Example 2.8 (continued). For the points in Example 2.3 we set

$$\begin{aligned}
h_1 &= & x_1 + z^{-3}x_2 + z^{-6}x_3\,, \\
h_2 &= & x_1 + z^{-5}x_2 + z^{-2}x_3\,, \\
h_3 &= & x_1 + \phantom{z^{-5}}x_2 + z^{-1}x_3\,, \text{ and} \\
h_4 &= z^{-1}x_1 + z^{-5}x_2 + \phantom{z^{-1}}x_3\,.
\end{aligned}$$

Then $h = h_1 \cdot h_2 \cdot h_3 \cdot h_4$ equals

$$h = z^{-1}x_1{}^4 + (z^{-6} + z^{-5} + z^{-4} + z^{-1})x_1{}^3x_2 + (z^{-7} + z^{-3} + z^{-2} + 1)x_1{}^3x_3 + (z^{-10} + z^{-9} + z^{-8} + z^{-6} + z^{-5} + z^{-4})x_1{}^2x_2{}^2 + (z^{-12} + z^{-11} + 3z^{-7} + 2z^{-6} + 2z^{-5} + 2z^{-3} + 1)x_1{}^2x_2x_3 + (z^{-9} + z^{-8} + z^{-6} + z^{-4} + z^{-2} + z^{-1})x_1{}^2x_3{}^2 + (z^{-13} + z^{-10} + z^{-9} + z^{-8})x_1x_2{}^3 + (z^{-16} + z^{-12} + 2z^{-11} + 2z^{-10} + z^{-9} + z^{-8} + z^{-7} + z^{-6} + z^{-5} + z^{-3})x_1x_2{}^2x_3 + (2z^{-13} + z^{-12} + z^{-11} + z^{-9} + z^{-8} + z^{-7} + 2z^{-6} + z^{-5} + z^{-4} + z^{-2})x_1x_2x_3{}^2 + (z^{-10} + z^{-8} + z^{-7} + z^{-3})x_1x_3{}^3 + z^{-13}x_2{}^4 + (z^{-16} + z^{-14} + z^{-10} + z^{-8})x_2{}^3x_3 + (z^{-17} + z^{-13} + 2z^{-11} + z^{-9} + z^{-5})x_2{}^2x_3{}^2 + (z^{-14} + z^{-12} + z^{-8} + z^{-6})x_2x_3{}^3 + z^{-9}x_3{}^4 \,.$$

Its max-tropicalization is

$$\mathrm{trop}^{\max}(h) = (-1 \odot x_1{}^{\odot 4}) \oplus (-1 \odot x_1{}^{\odot 3} \odot x_2) \oplus (x_1{}^{\odot 3} \odot x_3) \oplus (-4 \odot x_1{}^{\odot 2} \odot x_2{}^{\odot 2}) \oplus (x_1{}^{\odot 2} \odot x_2 \odot x_3) \oplus (-1 \odot x_1{}^{\odot 2}x_3{}^{\odot 2}) \oplus (-8 \odot x_1 \odot x_2{}^{\odot 3}) \oplus (-3 \odot x_1 \odot x_2{}^{\odot 2} \odot x_3) \oplus (-2 \odot x_1 \odot x_2 \odot x_3{}^{\odot 2}) \oplus (-3 \odot x_1 \odot x_3{}^{\odot 3}) \oplus (-13 \odot x_2{}^{\odot 4}) \oplus (-8 \odot x_2{}^{\odot 3} \odot x_3) \oplus (-5 \odot x_2{}^{\odot 2} \odot x_3{}^{\odot 2}) \oplus (-6 \odot x_2 \odot x_3{}^{\odot 3}) \oplus (-9 \odot x_3{}^{\odot 4}) \,.$$

The polynomial h and its max-tropicalization have 15 terms each, one for each maximal cell of the arrangement shown in Figure 1. For instance, the point $p = (0, 1, 0)$ is generic with fine type $T(p) = (12, 3, 4)$. Evaluating $\mathrm{trop}^{\max}(h)$ at p gives -1, and this maximum is attained for the unique tropical monomial $x_1{}^{\odot 2} \odot x_2 \odot x_3 = 2x_1 + x_2 + x_3$. We have $\mathbf{t}(p) = (2, 1, 1)$ for the coarse type.

Recall that a *composition* of n into d parts is a sequence $(t_1, t_2, \ldots, t_d) \in \mathbb{N}^d$ of nonnegative integers such that $t_1 + t_2 + \cdots + t_d = n$. Such compositions bijectively correspond to monomials of total degree n in d variables, of which there are exactly $\binom{n+d-1}{n} = \binom{n+d-1}{d-1}$.

Theorem 2.9. *For an arrangement $\mathcal{A} = \mathcal{A}(V)$, the map $\mathbf{t}$ from the set of cells in $\mathcal{C}_{\mathcal{A}}$ of maximal dimension $d-1$ to the set of compositions of n into d parts is injective. Moreover, if the points V are sufficiently generic then the map $\mathbf{t}$ is bijective.*

Proof. The injectivity of $\mathbf{t}$ follows immediately from Proposition 2.7.

Now suppose that the points in V are sufficiently generic. All coordinates are finite, and hence all linear monomials are present in the linear form h_i with non-zero coefficients. Since h is the product of $h_1, h_2, \ldots, h_n$ all possible monomials of degree n in the d indeterminates $x_1, x_2, \ldots, x_d$ actually occur in h. Therefore all tropical monomials occur in the tropicalization $\mathrm{trop}^{\max}(h)$. As we assumed that the coefficients in V are chosen sufficiently generic, the coefficients in the tropical polynomial $\mathrm{trop}^{\max}(h)$ are generic. Equivalently, each monomial of h defines a facet of $P_{\mathcal{A}}$, and hence the claim. □

Proposition 4.2 provides an interpretation of the above result in terms of mixed subdivisions of dilated simplices.

Corollary 2.10. *For an arrangement $\mathcal{A} = \mathcal{A}(V)$, the number of cells in $\mathcal{C}_{\mathcal{A}}$ of maximal dimension $d-1$ does not exceed $\binom{n+d-1}{n}$. If the points V are sufficiently generic then this bound is attained.*

Note that the injectivity of $\mathbf{t}$ does not extend to the lower-dimensional cells, even in the sufficiently generic case. For instance, in the arrangement considered in Examples 2.3 and 2.8 the points $(0, 2, 3)$ and $(0, 3, 5)$ lie in the relative interiors of two distinct 1-cells; yet they share the same coarse type $(1, 1, 3)$. However coarse types are *locally* distinct in the following sense.

Proposition 2.11. *Let C and D be two distinct cells in $\mathcal{C}_{\mathcal{A}}$. If C is contained in the closure of D then C and D have distinct coarse types.*

Proof. By [7, Cor. 13] we have $T_{\mathcal{A}}(D) \leq T_{\mathcal{A}}(C)$ and $T_{\mathcal{A}}(D) \neq T_{\mathcal{A}}(C)$. This, in particular implies that C is contained in more closed sectors with respect to $\mathcal{A}$ and hence C and D cannot have the same coarse type. $\qquad\square$

Note also that the boundedness of a cell in $\mathcal{C}_{\mathcal{A}}$ can be read off its coarse type.

Proposition 2.12 ([7, Cor. 12]). *Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement and let $C \in \mathcal{C}_{\mathcal{A}}$ be a cell in the induced decomposition. Then C is bounded if and only if $\mathbf{t}(C)_i > 0$ for all $i = 1, \ldots, n$.*

Proof. We noted previously that for two points $p, q \in \mathbb{T}^{d-1}$ we have $p \in S_k^{\max}(q)$ if and only if $q \in S_k^{\min}(p)$. Now, the point p is contained in an unbounded cell of $\mathcal{C}_{\mathcal{A}}$ if there is a k such that $S_k^{\min}(p) \cap V = \emptyset$. This is the case if and only if $t_{\mathcal{A}}(p)_k = 0$, that is, there is no hyperplane H_i for which p is contained in the k-th sector. $\qquad\square$

Remark 2.13. There is another perspective on Proposition 2.7 that we wish to share. The standard valuation $\nu : (\mathbb{C}\{\{z\}\}^{\mathrm{gen}})^* \to \mathbb{R}$ on the field of (generalized) Puiseux series maps an element to its order. For $\omega \in \mathbb{R}^d$, the ω-*weight* of $f = \sum_a c_a \mathbf{x}^a \in \mathbb{C}\{\{z\}\}^{\mathrm{gen}}[x_1, \ldots, x_d]$ is $\omega(f) = \max\{\nu(c_a) + \langle \omega, a \rangle : c_a \neq 0\}$ and the *initial form* of the polynomial f is given by $\mathrm{in}_\omega(f) := \sum\{\mathbf{x}^a : \nu(c_a) + \langle \omega, a \rangle = \omega(f)\}$. Finally, for an ideal $I \subset \mathbb{C}\{\{z\}\}^{\mathrm{gen}}[x_1, \ldots, x_d]$ the *initial ideal* $\mathrm{in}_\omega(I)$ is generated by all $\mathrm{in}_\omega(f)$ for $f \in I$. An alternative characterization of the tropical variety was given by Speyer and Sturmfels [30, Thm. 2.1] as the collection of weights $\omega \in \mathbb{R}^d$ such that $\mathrm{in}_\omega(I)$ does not contain a monomial. For a tropical hyperplane arrangement $\mathcal{A} = \mathcal{A}(V)$ with defining polynomial $h = h_{\mathcal{A}} \in \mathbb{C}\{\{z\}\}^{\mathrm{gen}}[x_1, \ldots, x_d]$ the coarse type $\mathbf{t}(p)$ of a generic point $p \in \mathbb{T}^{d-1} \setminus \mathcal{A}$ satisfies

$$\langle \mathbf{x}^{\mathbf{t}(p)} \rangle = \mathrm{in}_p(\langle h \rangle).$$

Remark 2.14. The evaluation of the tropical polynomial $\mathrm{trop}^{\max}(h)$ at some point p can be modeled as a min-cost-flow problem as follows: Define a directed graph with nodes α, $\beta_1, \beta_2, \ldots, \beta_n$, $\gamma_1, \gamma_2, \ldots, \gamma_d$, and δ. For each $i \in [n]$ and each $k \in [d]$ there are the following arcs: from α to β_i with cost 0 and capacity 1, from β_i to γ_k with cost v_{ik} and capacity 1, from γ_k to δ with cost p_k and unbounded capacity. Now $\mathrm{trop}^{\max}(h)(p)$ equals the minimal cost of shipping n units of flow from the unique source α to the unique sink δ. This problem has a strongly polynomial time solution in the parameters n and d; see [28, Chapter 12]. This is remarkable in view of the fact that the tropical polynomial $\mathrm{trop}^{\max}(h)$ may have exponentially many monomials.

2.3. **Topology of types.** We next investigate topological properties of certain subsets of $\mathbb{T}^{d-1}$ induced by a tropical hyperplane arrangement. The motivation for such a study comes from our applications to (co)cellular resolutions as described in Section 3. However, since the methods used to establish the desired properties are based on notions from (coarse) tropical convexity, we include them here. In particular, we show that certain subsets of $\mathbb{T}^{d-1}$ are contractible by proving that they are, in fact, tropically convex. Let us emphasize that none of the results that follow require the hyperplanes to be in general position.

We begin with the following observation, which was established in [7, Thm. 2]. We include a short proof for the sake of completeness.

Proposition 2.15. *A tropically convex set is contractible.*

Proof. The *distance* of two points $p, q \in \mathbb{T}^{d-1}$ is defined as

$$\mathrm{dist}(p, q) := \max_{1 \leq i < j \leq d} |p_i - p_j + q_j - q_i|.$$

We have the equations

$$\big(\operatorname{dist}(p,q) \odot p\big) \oplus q \;=\; q \quad \text{and} \quad p \oplus \big(\operatorname{dist}(p,q) \odot q\big) \;=\; p\,.$$

Now let p be a point in some tropically convex set S. The map

$$\eta \,:\, S \times [0,1] \to S \,:\, (q,t) \mapsto \big(((1-t) \cdot \operatorname{dist}(p,q)) \odot p\big) \oplus \big((t \cdot \operatorname{dist}(p,q)) \odot q\big)$$

is continuous, and it contracts the set S to the point p. $\qquad\square$

Recall that for two types $T, T' \in \{0,1\}^{n \times d}$ we write $T \leq T'$ for the componentwise induced partial order. We let $\min(T, T')$ and $\max(T, T')$ denote the tables with entries given by the componentwise minimum and maximum, respectively.

Proposition 2.16. *Let $\mathcal{A} = \mathcal{A}(V)$ be a* max*-tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and $p, q \in \mathbb{T}^{d-1}$. Then*

$$\min(T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)) \;\leq\; T_{\mathcal{A}}(r) \;\leq\; \max(T_{\mathcal{A}}(p), T_{\mathcal{A}}(q))$$

for every point $r \in \operatorname{tconv}^{\max}\{p,q\}$ on the max*-tropical line segment between p and q.*

Proof. Let $r = (\lambda \odot p) \oplus (\mu \odot q) = \max\{\lambda\mathbb{1} + p, \mu\mathbb{1} + q\}$ for $\lambda, \mu \in \mathbb{R}$ and let $k \in [d]$ be arbitrary but fixed. We treat each inequality separately but in each case we assume without loss of generality that $r_k = \lambda + p_k \geq \mu + q_k$.

For the first inequality suppose that both p and q are in the k-th sector of some hyperplane $H(v)$, so that $p_k - p_i \geq v_k - v_i$ and $q_k - q_i \geq v_k - v_i$ for all $i \in [d]$. Now, for $j \in [d]$ we distinguish two cases. If $r_j = \lambda + p_j \geq \mu + q_j$, then $r_j - r_k = p_j - p_k$, so that r is in the k-th sector of $H(v)$. If $r_j = \mu + q_j \geq \lambda + p_j$, then $r_k - r_j \geq \mu + q_k - r_j = \mu + q_k - (\mu + q_j) = q_k - q_j$. Hence $r_k - r_j \geq v_k - v_j$ for all $j \in [d]$, and we conclude that r is in the k-th sector of $H(v)$.

For the second inequality suppose that r is contained in the k-th sector of some hyperplane $H(v)$, so that $r_k - r_j \geq v_k - v_j$ for all $j \in [d]$. Since $r_k = \lambda + p_k \geq \mu + q_k$ we have that $\lambda + p_k \geq v_k - v_j + r_j$ for all $j \in [d]$. Also, $r_j \geq \lambda + p_j$ and hence $\lambda + p_k \geq v_k - v_j + \lambda + p_j$. We conclude $p_k - p_j \geq v_k - v_j$ for all $j \in [d]$. Hence p is in the k-th sector of $H(v)$, as desired. $\quad\square$

From the definition the coarse type we obtain the following statement regarding coarse types.

Corollary 2.17. *Let $\mathcal{A} = \mathcal{A}(V)$ be a* max*-tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and $p, q \in \mathbb{T}^{d-1}$. Then*

$$\mathbf{t}_{\mathcal{A}}(r) \;\leq\; \max(\mathbf{t}_{\mathcal{A}}(p), \mathbf{t}_{\mathcal{A}}(q))$$

for $r \in \operatorname{tconv}^{\max}\{p,q\}$.

From Proposition 2.16 and Corollary 2.17 we obtain the following result regarding the topology of regions of bounded fine and coarse (co)type. These will be the main tools for establishing results regarding (co)cellular resolutions in Section 3.

Corollary 2.18. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of n* max*-tropical hyperplanes in $\mathbb{T}^{d-1}$, and let $B \in \{0,1\}^{n \times d}$ and $\mathbf{b} \in \mathbb{N}^d$. With labels determined by fine (respectively, coarse) type the following subsets of $\mathbb{T}^{d-1}$ are* max*-tropically convex and hence contractible:*

$$
\begin{aligned}
(\mathcal{C}_{\mathcal{A}}, T)_{\leq B} &:= \big\{ p \in \mathbb{T}^{d-1} \,:\, T_{\mathcal{A}}(p) \leq B \big\} \;=\; \bigcup \big\{ C \in \mathcal{C}_{\mathcal{A}} \,:\, T_{\mathcal{A}}(C) \leq B \big\} \\
(\mathcal{C}_{\mathcal{A}}, \mathbf{t})_{\leq \mathbf{b}} &:= \big\{ p \in \mathbb{T}^{d-1} \,:\, \mathbf{t}_{\mathcal{A}}(p) \leq \mathbf{b} \big\} \;=\; \bigcup \big\{ C \in \mathcal{C}_{\mathcal{A}} \,:\, \mathbf{t}_{\mathcal{A}}(C) \leq \mathbf{b} \big\}\,.
\end{aligned}
$$

Similarly, with labels determined by fine (respectively, coarse) cotype the following subsets of $\mathbb{T}^{d-1}$ *are* min-*tropically convex and hence contractible:*

$$
\begin{aligned}
(\mathcal{C}_\mathcal{A}, \overline{T})_{\leq B} &:= \left\{ p \in \mathbb{T}^{d-1} : \overline{T}_\mathcal{A}(p) \leq B \right\} = \bigcup \left\{ C \in \mathcal{C}_\mathcal{A} : \overline{T}_\mathcal{A}(C) \leq B \right\} \\
(\mathcal{C}_\mathcal{A}, \overline{\mathbf{t}})_{\leq \mathbf{b}} &:= \left\{ p \in \mathbb{T}^{d-1} : \overline{\mathbf{t}}_\mathcal{A}(p) \leq \mathbf{b} \right\} = \bigcup \left\{ C \in \mathcal{C}_\mathcal{A} : \overline{\mathbf{t}}_\mathcal{A}(C) \leq \mathbf{b} \right\}.
\end{aligned}
$$

As a consequence, the two subsets of $\mathbb{T}^{d-1}$ *obtained by replacing the complex* $\mathcal{C}_\mathcal{A}$ *in the above pair of formulas with the bounded complex* $\mathcal{B}_\mathcal{A}$ *are* min-*tropically convex and hence contractible.*

Proof. The max-tropical convexity of $(\mathcal{C}_\mathcal{A}, T)_{\leq B}$ follows from Proposition 2.16, and Corollary 2.17 establishes the same property for the coarse variant $(\mathcal{C}_\mathcal{A}, \mathbf{t})_{\leq \mathbf{b}}$.

If r is a point in the min-tropical line segment between p and q a reasoning similar to the proof of Proposition 2.16 shows that $T_\mathcal{A}(r) \geq \min(T_\mathcal{A}(p), T_\mathcal{A}(q))$. Passing to complements yields $\overline{T}_\mathcal{A}(r) \leq \max(\overline{T}_\mathcal{A}(p), \overline{T}_\mathcal{A}(q))$ and thus $\overline{\mathbf{t}}_\mathcal{A}(r) \leq \max(\overline{\mathbf{t}}_\mathcal{A}(p), \overline{\mathbf{t}}_\mathcal{A}(q))$ for the coarse types. We conclude that the sets $(\mathcal{C}_\mathcal{A}, \overline{T})_{\leq B}$ and $(\mathcal{C}_\mathcal{A}, \overline{\mathbf{t}})_{\leq \mathbf{b}}$ are min-tropically convex. For the last claim, we note that the tropical complex $\mathcal{B}_\mathcal{A}$ itself is min-tropically convex; in view of Proposition 2.12 it is a down-set of $\mathcal{C}_\mathcal{A}$ under the cotype labeling. Hence both $(\mathcal{B}_\mathcal{A}, \overline{T})_{\leq B}$ and $(\mathcal{B}_\mathcal{A}, \overline{\mathbf{t}})_{\leq \mathbf{b}}$ are intersections of min-tropically convex sets. $\qquad\square$

Note that the sets of bounded (coarse) type need not be closed or bounded. In Figure 2 we illustrate an example of a coarse down-set from our running example.
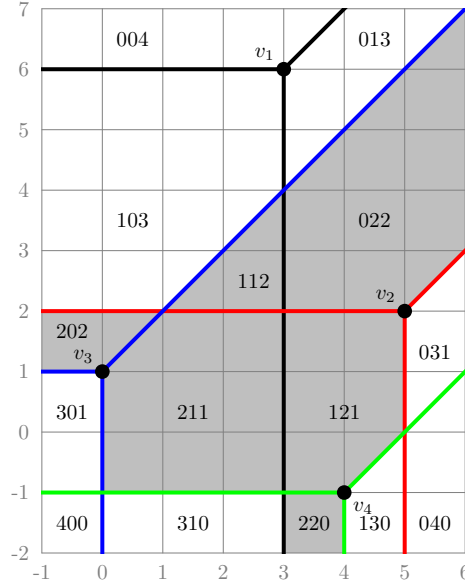

FIGURE 2. Coarse down-set $(C_\mathcal{A})_{\leq(2,2,2)}$ for the arrangement $\mathcal{A}$ from Example 2.3.

3. RESOLUTIONS

In this section we show how the polyhedral complexes $\mathcal{C}_\mathcal{A}$ and $\mathcal{B}_\mathcal{A}$ arising from a tropical hyperplane arrangement $\mathcal{A} = \mathcal{A}(V)$ support resolutions for associated monomial ideals. We begin with a few definitions.

Definition 3.1. Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$. The *fine type* and *fine cotype ideal* associated to $\mathcal{A}$ are the squarefree monomial ideals

$$I_{T(\mathcal{A})} = \langle \mathbf{x}^{T(p)} : p \in \mathbb{T}^{d-1} \rangle \subset \mathbb{k}[x_{11}, x_{12}, \ldots, x_{nd}]$$

$$I_{\overline{T}(\mathcal{A})} = \langle \mathbf{x}^{\overline{T}(p)} : p \in \mathbb{T}^{d-1} \rangle \subset \mathbb{k}[x_{11}, x_{12}, \ldots, x_{nd}]$$

where $\mathbf{x}^{T(p)} = \prod\{x_{ij} : T(p)_{ij} = 1\}$. Analogously, the *coarse type* and *coarse cotype ideal* associated to $\mathcal{A}$ are given by

$$I_{\mathbf{t}(\mathcal{A})} = \langle \mathbf{x}^{\mathbf{t}(p)} : p \in \mathbb{T}^{d-1} \rangle \subset \mathbb{k}[x_1, x_2, \ldots, x_d]$$

$$I_{\overline{\mathbf{t}}(\mathcal{A})} = \langle \mathbf{x}^{\overline{\mathbf{t}}(p)} : p \in \mathbb{T}^{d-1} \rangle \subset \mathbb{k}[x_1, x_2, \ldots, x_d].$$

where $\mathbf{x}^{\mathbf{t}(p)} = x_1^{t_1} x_2^{t_2} \cdots x_d^{t_d}$ with $\mathbf{t}(p) = (t_1, t_2, \ldots, t_d)$.

An analogous construction of fine cotype ideals for classical hyperplane arrangements first appears in Novik et al. [22] in the form of *oriented matroid ideals*, where monomial generators are given by (complements of) covector data. As remarked earlier, the fine type of a tropical hyperplane arrangement can be thought of as the covector data of a tropical oriented matroid and, from this perspective, a tropical analog of [22] is given in [4]. In Sections 3 and 6 we will establish further connections between our work and the results and constructions of [4]. A common generalization in the form of fan arrangements will be given in [27]. In [22] the authors also consider *matroid ideals* given by specializing the underlying oriented matroid ideal, in effect not distinguishing between different sides of a (classical) hyperplane. We point out in the case of our coarse type ideals, the covector data coming from the different *sectors* induced by a (tropical) hyperplane (the tropical analog of the 'side') is maintained, whereas the *labels* on the hyperplanes are no longer distinguished.

3.1. **Cellular and cocellular resolutions.** The relation between the decompositions of $\mathbb{T}^{d-1}$ and the various ideals of Definition 3.1 is given via (co)cellular resolutions. Whereas cellular resolutions are by now a standard tool in (combinatorial) commutative algebra, *co*cellular resolutions seem to be less popular. As they arise naturally in connection with cotypes, we take the opportunity to discuss cocellular resolutions alongside cellular resolutions in some detail. Our presentation is based on the book [21] of Miller and Sturmfels.

For a fixed field $\mathbb{k}$ we let $S = \mathbb{k}[x_1, \ldots, x_m]$ be the polynomial ring equipped with the $\mathbb{Z}^m$-grading given by $\deg \mathbf{x}^{\mathbf{a}} = \mathbf{a} \in \mathbb{Z}^m$. A free $\mathbb{Z}^m$-graded resolution $\mathcal{F}_\bullet$ of a $\mathbb{Z}^m$-graded module M is an algebraic complex of $\mathbb{Z}^m$-graded S-modules

$$\mathcal{F}_\bullet : \quad \cdots \xrightarrow{\phi_{k+1}} F_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to 0$$

where $F_i \cong \bigoplus_{\mathbf{a} \in \mathbb{Z}^m} S(-\mathbf{a})^{\beta_{i,a}}$ are free $\mathbb{Z}^m$-graded S-modules, the maps ϕ_i are homogeneous, and such that the complex is exact except for $\operatorname{coker} \phi_1 \cong M$. The resolution is called *minimal* exactly when $\beta_{i,a} = \dim_\mathbb{k} \operatorname{Tor}_i^S(S/I, \mathbb{k})_a$ and the numbers $\beta_{i,a}$ are called the *fine graded Betti numbers*.

An efficient way of encoding $\mathbb{Z}^m$-graded resolutions of monomial ideals is given by *cellular* and *cocellular resolutions*, introduced in [2] and [20], respectively. Let $\mathcal{P}$ be an oriented polyhedral complex and let $(\mathbf{a}_H)_{H \in \mathcal{P}} \in \mathbb{Z}^m$ be a labeling of the cells of $\mathcal{P}$ such that

$$\mathbf{a}_H = \max\{\mathbf{a}_G : \text{ for } G \subset H \text{ a face}\}.$$

The labeled complex $(\mathcal{P}, \mathbf{a})$ gives rise to an algebraic complex of free $\mathbb{Z}^m$-graded S-modules in the following way: Let $(C_\bullet, \partial_\bullet)$ be the cellular chain complex for $\mathcal{P}$ and for two cells

$G, H \in \mathcal{P}$ with $\dim H = \dim G + 1$ denote by $\varepsilon(H, G) \in \{0, \pm 1\}$ the coefficient of G in the cellular boundary of H. Now define free modules

$$F_i := \bigoplus_{H \in \mathcal{P}, \, \dim H = i+1} S(-\mathbf{a}_H).$$

The differentials $\phi_i : F_i \to F_{i-1}$ are given on generators by

$$\phi_i(e_H) := \sum_{\dim G = \dim H - 1} \varepsilon(H, G) \mathbf{x}^{\mathbf{a}_H - \mathbf{a}_G} e_G.$$

It can be verified that this defines an algebraic complex $\mathcal{F}_\bullet^{\mathcal{P}}$. For $\mathbf{b} \in \mathbb{Z}^m$ denote by $\mathcal{P}_{\leq \mathbf{b}}$ the subcomplex given by all cells $H \in \mathcal{P}$ with $\mathbf{a}_H \leq \mathbf{b}$, that is $(\mathbf{a}_H)_i \leq b_i$ for all $i \in [m]$.

Lemma 3.2 ([21, Prop. 4.5]). *Let $\mathcal{F}_\bullet^{\mathcal{P}}$ be the algebraic complex obtained from the labeled polyhedral complex $(\mathcal{P}, \mathbf{a})$. If for every $\mathbf{b} \in \mathbb{Z}^n$ the subcomplex $\mathcal{P}_{\leq \mathbf{b}}$ is acyclic over $\Bbbk$, then $\mathcal{F}_\bullet^{\mathcal{P}}$ resolves the quotient of S by the ideal $\langle \mathbf{x}^{\mathbf{a}_v} : v \in P \text{ vertex} \rangle$. Furthermore, the resolution is minimal if $\mathbf{a}_H \neq \mathbf{a}_G$ for any two faces $G \subset H$ with $\dim H = \dim G + 1$.*

The complex $\mathcal{F}_\bullet^{\mathcal{P}}$ is called a *cellular resolution* if it meets the criterion above, and we say that the polyhedral complex $\mathcal{P}$ *supports* the resolution. If the labeling is such that

$$\mathbf{a}_H = \max \{ \mathbf{a}_G : \text{ for } G \supset H \text{ a face} \}$$

then $\mathcal{P}$ is said to be *colabeled* and gives rise to an algebraic complex utilizing the cellular *cochain* complex of $\mathcal{P}$. For this, let $\mathcal{F}_\mathcal{P}^\bullet$ denote the algebraic complex with free S-modules $F^i := F_i$ as defined above and differentials $\phi^i : F^{i-1} \to F^i$ with

$$\phi^i(e_H) := \sum_{\dim G = \dim H + 1} \delta(H, G) \mathbf{x}^{\mathbf{a}_H - \mathbf{a}_G} e_G$$

where $\delta(H, G)$ records the corresponding coefficient in the coboundary map for $\mathcal{P}$. If the algebraic complex is acyclic, the resulting resolution is called *cocellular*. For $\mathbf{b} \in \mathbb{Z}^m$ the collection $\mathcal{P}_{\leq \mathbf{b}}$ of relatively open cells H with $\mathbf{a}_H \leq \mathbf{b}$ is *not* a subcomplex. However, as a topological space it is the union of the relatively open stars of cells G for which $\mathbf{a}_G \leq \mathbf{b}$ is minimal and the cochain complex of the nerve is isomorphic to the degree $\mathbf{b}$ component of $\mathcal{F}_\mathcal{P}^\bullet$. This yields an analogous criterion regarding the exactness of $\mathcal{F}_\mathcal{P}^\bullet$. The proof of Lemma 3.2 given in [21] essentially proves the following criterion.

Lemma 3.3. *If $\mathcal{P}_{\leq \mathbf{b}}$ is acyclic over $\Bbbk$ for every $\mathbf{b} \in \mathbb{Z}^n$ then $\mathcal{F}_\mathcal{P}^\bullet$ resolves S/I where $I = \langle \mathbf{x}^{\mathbf{a}_H} : H \in \mathcal{P} \text{ maximal cell} \rangle$. The resolution is minimal if $\mathbf{a}_H \neq \mathbf{a}_G$ for any two faces $G \subset H$ with $\dim H = \dim G + 1$.*

3.2. **Resolutions from the arrangement.** As usual let $\mathcal{A} = \mathcal{A}(V)$ be a max-tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and let $\mathcal{C}_\mathcal{A}$ be the induced polyhedral decomposition of $\mathbb{T}^{d-1}$. As we have seen, every cell in $\mathcal{C}_\mathcal{A}$ is naturally assigned a matrix and a vector determined by its fine and coarse type, respectively. The next result states that these assignments are actually *colabelings* in the sense of Section 3.1.

Proposition 3.4. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of tropical hyperplanes in $\mathbb{T}^{d-1}$. For every cell $C \in \mathcal{C}_\mathcal{A}$ of codimension ≥ 1 we have*

$$T_\mathcal{A}(C) = \max\{T_\mathcal{A}(D) : C \subset D\}, \text{ and}$$
$$\mathbf{t}_\mathcal{A}(C) = \max\{\mathbf{t}_\mathcal{A}(D) : C \subset D\}.$$

Thus, both the fine type and the coarse type yield a colabeling for the complex $\mathcal{C}_\mathcal{A}$.

Proof. As we remarked before $C \subseteq D$ implies $T_{\mathcal{A}}(D) \leq T_{\mathcal{A}}(C)$ and thus we have to show that $T_{\mathcal{A}}(C)$ is not strictly larger. For $k \in [d]$ consider the set $Q_k \subset \mathbb{R}^d$ given by all points $x \in \mathbb{R}^d$ such that $x_k \leq x_i$ for all $i \neq k$. The set Q_k is an ordinary polyhedron of dimension d and it can be checked that for $p \in \mathbb{T}^{d-1}$ we have that $p \in S_k^{\max}(v_i)$ implies $p + Q_k \subseteq S_k(v_i)$. Now, since $\mathrm{codim}(C) \geq 1$ we get that $C + Q_k$ meets the star of C thereby showing that if $T_{\mathcal{A}}(C)_{ik} = 1$ for some i, then there is a cell $D \supset C$ with $T_{\mathcal{A}}(D)_{ik} = 1$.

The same argument proves the purported equality for the coarse type. $\qquad\square$

As an immediate consequence we obtain sets of generators for the fine and the coarse type ideals.

Corollary 3.5. *For a tropical hyperplane arrangement $\mathcal{A}$ both the fine and coarse type ideals are generated by monomials corresponding to the respective types on the inclusion-maximal cells of $\mathcal{C}_{\mathcal{A}}$.*

We now have all the ingredients to establish our first main result regarding the relation between fine/coarse type ideals and the polyhedral decomposition $\mathcal{C}_{\mathcal{A}}$.

Theorem 3.6. *Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement, and let $\mathcal{C}_{\mathcal{A}}$ be the decomposition of the tropical torus $\mathbb{T}^{d-1}$ induced by $\mathcal{A}$. Then with labels given by fine type (respectively, coarse type) the labeled complex $\mathcal{C}_{\mathcal{A}}$ supports a minimal cocellular resolution of the fine type ideal $I_{T(\mathcal{A})}$ (respectively, the coarse type ideal $I_{\mathbf{t}(\mathcal{A})}$).*

Proof. By Proposition 3.4 the polyhedral complex $\mathcal{C}_{\mathcal{A}}$ is colabeled by both fine and coarse type. It follows from Lemma 3.3 and Corollary 2.18 that this yields a cellular resolution of the respective type ideal. The minimality is a consequence of Proposition 2.11. $\qquad\square$

A key point is that all of the above is valid for point configurations V which are not necessarily in general position. In the case of hyperplanes in general position, the coarse type ideal is well known and, in particular, is independent of the choice of hyperplanes.

Corollary 3.7. *Let $\mathcal{A} = \mathcal{A}(V)$ be a* sufficiently generic *arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$. Then $\mathcal{C}_{\mathcal{A}}$ supports a minimal cocellular resolution of*

$$I_{\mathbf{t}(\mathcal{A})} \;=\; \langle x_1, \dots, x_d \rangle^n$$

the n-th power of the homogeneous maximal ideal.

Since the minimal resolution of an ideal is unique up to isomorphism, we have that the resolution arising from Corollary 3.7 is always isomorphic (as a chain complex), to the well-known Eliahou-Kervaire resolution [21, §2.3]. However, Corollary 3.7 shows that there is a multitude of colabeled complexes coming from tropical hyperplane arrangements that give rise to a cellular description of the minimal resolution of $\langle x_1, \dots, x_n \rangle^d$.

We next turn to the other class of ideals introduced in the beginning of this section, namely the fine and coarse *co*type ideals. As was the case with the type ideals, we first consider the labeling of the relevant complexes.

Proposition 3.8. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of tropical hyperplanes. For every cell $D \in \mathcal{C}_{\mathcal{A}}$ of dimension ≥ 1 we have*

$$\overline{T}_{\mathcal{A}}(D) \;=\; \max\{\overline{T}_{\mathcal{A}}(C) : C \subset D\}, \;\; and$$
$$\overline{\mathbf{t}}_{\mathcal{A}}(D) \;=\; \max\{\overline{\mathbf{t}}_{\mathcal{A}}(C) : C \subset D\}.$$

Thus, both the fine and coarse types yield labelings for the complex $\mathcal{C}_{\mathcal{A}}$.

Proof. The assertion follows by showing that

$$T_{\mathcal{A}}(D) \; \geq \; \min\{T_{\mathcal{A}}(C) : C \subset D\}\,.$$

We mimic the argument in the proof of Proposition 3.4. It follows from the full-dimensionality of Q_k that $C + Q_k$ intersects D for every cell $C \subset D$ and thus $T(D)_{ik} \geq T(C)_{ik}$ for all C in the boundary of D. $\qquad\square$

The previous proposition implies that the cotype ideals are generated by the inclusion-*minimal* faces (namely, the 0-dimensional cells) of $\mathcal{C}_{\mathcal{A}}$. As a consequence (together with the last part of Corollary 2.18) we see that resolutions of these ideals are supported on the collection of bounded faces of $\mathcal{C}_{\mathcal{A}}$, and we obtain our next main result. This generalizes [4, Theorem 1].

Theorem 3.9. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of tropical hyperplanes and let $\mathcal{B}_{\mathcal{A}}$ be the subcomplex of bounded cells of $\mathcal{C}_{\mathcal{A}}$. Then $\mathcal{B}_{\mathcal{A}}$, with labels given by fine cotype (respectively, coarse cotype) supports a minimal resolution of the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$ (respectively, coarse cotype ideal $I_{\overline{\mathfrak{t}}(\mathcal{A})}$).*

Remark 3.10. For fixed n and d we have a ring map $\psi : \Bbbk[x_{11}, \dots, x_{nd}] \to \Bbbk[x_1, \dots, x_d]$ determined by $x_{ij} \mapsto x_j$. This map takes fine (co)type ideals to coarse (co)type ideals. In either case, both the fine and the coarse ideals are resolved by the same polyhedral complex, and one might hope that the resolution of the fine ideal descends to the coarse ideal via ψ. Such an approach is employed in [22] in the passage from the oriented matroid to the matroid ideal. There it is shown that any oriented matroid ideal J is Cohen-Macaulay, with $(x_{11}-x_{21}, x_{21}-x_{22}, \dots, x_{n1}-x_{n2})$ forming a linear system of parameters (and hence a regular sequence) in S/J. Unfortunately, this approach does *not* work in our context. It turns out that the fine cotype ideals $I_{\overline{T}(\mathcal{A})}$ are not in general Cohen-Macaulay. Indeed, regarding $I_{\overline{T}(\mathcal{A})}$ as a Stanley-Reisner ideal we see that $S/I_{\overline{T}(\mathcal{A})}$ has dimension $dn - 2$, whereas an application of the Auslander-Buchsbaum formula (and knowledge of the minimal resolution) shows that its depth is $dn - d$. The coarsening sequence $\{x_{11} - x_{21}, \dots, x_{11} - x_{n1}, \dots, x_{1d} - x_{nd}\}$ is also of length $dn - d$, but one can show that it is not a regular sequence (not even a system of parameters).

4. The mixed subdivision picture

As mentioned in Section 2.2, an arrangement $\mathcal{A} = \mathcal{A}(V)$ of n hyperplanes in $\mathbb{T}^{d-1}$ gives rise to a regular subdivision of the product of two simplices $\Delta_{n-1} \times \Delta_{d-1}$. To see this, we let $\Delta_{n-1} \subset \mathbb{R}^n$ denote the convex hull of the standard basis vectors $\{e_i : 1 \leq i \leq n\}$. If $V = (v_1, \dots, v_n)$ is the ordered sequence of the apices of the arrangement $\mathcal{A}$, we lift each vertex (e_i, e_j) of $\Delta_{n-1} \times \Delta_{d-1}$ to the height $(v_i)_j$, the j-th coordinate of the i-th apex. Taking the lower convex hull of the resulting configuration of points induces a (by definition regular) subdivision of $\Delta_{n-1} \times \Delta_{d-1}$. A main result from [7] is that the combinatorial types of the tropical complexes generated by a set of n vertices in $\mathbb{T}^{d-1}$, seen as polytopal complexes, are in bijection with the regular polyhedral subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$.

At this point, the reader may find it useful to contemplate the schematic diagram in Figure 3 below. It sketches how various relevant combinatorial concepts are related. Again V denotes our ordered sequence of n points in $\mathbb{T}^{d-1}$, which can be thought of as the apices of a max hyperplane arrangement $\mathcal{A}$ (where the hyperplane with apex v_i corresponds to the tropical vanishing of the form $-v_i \odot x$). We have $\mathcal{B}_{\mathcal{A}}$ given by the min-tropical convex hull $\mathrm{tconv}^{\min}(V)$, which is the bounded part of the polyhedral complex $\mathcal{C}_{\mathcal{A}}$ determined by

the coarse types of the arrangement $\mathcal{A}$. The correspondence described above (here denoted 'DS') associates to $\mathcal{B}_{\mathcal{A}}$ a certain regular polyhedral subdivision $\Delta_{\mathcal{A}}$ of $\Delta_{n-1} \times \Delta_{d-1}$; this subdivision is dual to the complex $\mathcal{C}_{\mathcal{A}}$. The Cayley trick (described in more detail below, here denoted 'Cay') then associates to that regular subdivision a regular *mixed* subdivision $\Sigma_{\mathcal{A}}$ of the dilated simplex $n\Delta_{d-1}$. Moreover, we have the 'switch' isomorphism obtained by exchanging coordinates of the product; applying the Cayley trick then gives a regular mixed subdivision $\Sigma_{\mathcal{A}}^{*}$ of $d\Delta_{n-1}$. Finally, observe that $n\Delta_{d-1}$ is the Newton polytope of the polynomial h defined in (1). The mixed subdivision $\Sigma_{\mathcal{A}}$ is the *privileged* regular subdivision of $n\Delta_{d-1}$ defined by the coefficients of $\mathrm{trop}^{\max}(h)$. The corresponding arrow in the diagram is marked 'New' for 'Newton'.

$$V$$

$$\mathcal{B}_{\mathcal{A}} = \mathrm{tconv}^{\min}(V) \qquad \text{New}$$

$$\text{bounded} \qquad \qquad \text{DS}$$

$$\mathcal{C} = \mathcal{C}^{\max}(V) \xrightarrow{\text{dual}} \Delta_{\mathcal{A}} \text{ of } \Delta_{n-1} \times \Delta_{d-1} \xrightarrow{\text{Cay}} \Sigma_{\mathcal{A}} \text{ of } n\Delta_{d-1}$$

$$\cong \Big\downarrow \text{switch}$$

$$\Delta_{\mathcal{A}} \text{ of } \Delta_{d-1} \times \Delta_{n-1} \xrightarrow[\text{Cay}]{} \Sigma_{\mathcal{A}}^{*} \text{ of } d\Delta_{n-1}$$
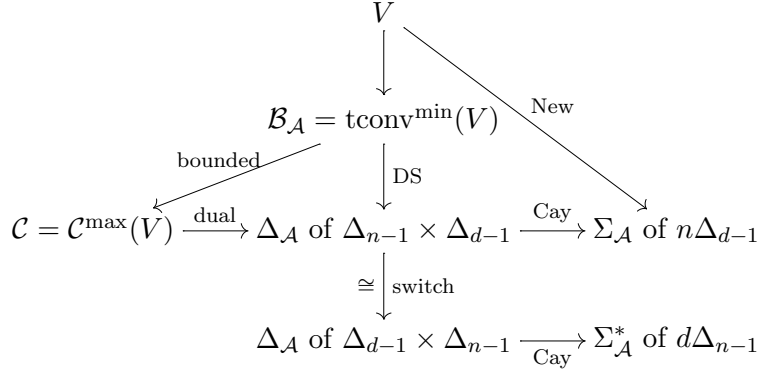
FIGURE 3. Diagram explaining how the various combinatorial concepts are related.

4.1. **The Cayley trick and mixed subdivisions.** The *Cayley trick* gives a bijection between the polyhedral subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$ and the *mixed* subdivisions of $n\Delta_{d-1}$. More generally, the Cayley trick relates mixed subdivisions of the Minkowski sum of several polytopes $P_1, P_2, \ldots, P_k$ with the polyhedral subdivisions of a certain polytope known as the *Cayley embedding* of those polytopes. We refer to [26] for a good account of the Cayley trick in the context of products of simplices, but we wish to discuss some of the basics here. This way it will also become apparent in which way products of simplices arise. The collection of all polyhedral subdivisions of a polytope P form a poset under refinement, and the minimal elements are the triangulations of P.

Recall that if $P_1, P_2, \ldots, P_n$ are polytopes in $\mathbb{R}^d$, then the *Minkowski sum* is defined to be the polytope

$$P_1 + P_2 + \cdots + P_n := \{x_1 + x_2 + \cdots + x_n : x_i \in P_i\}.$$

If $P = P_1 + P_2 + \cdots + P_n$ is a Minkowski sum of polytopes, a *mixed cell* $B \subseteq P$ is defined to be a Minkowski sum $B_1 + \cdots + B_n$, where each B_i is a polytope with vertices among those of P_i. A *mixed subdivision* of P is then a subdivision of P consisting of mixed cells. Once again, the mixed subdivisions of P form a poset under refinement, and in this case the minimal elements are called *fine* mixed subdivisions.

Now, if as above $P_1, P_2, \ldots, P_n$ are polytopes in $\mathbb{R}^d$, we again let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and let $\iota_i : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^n$ denote the inclusion $x \mapsto (x, e_i)$. The *Cayley embedding* of $P_1, P_2, \ldots, P_n$ is defined to be

$$\mathrm{Cayley}(P_1, P_2, \ldots, P_n) := \mathrm{conv}\left(\bigcup \iota_i(P_i)\right),$$

the convex hull of all these inclusions. The intersection of $\mathrm{Cayley}(P_1, P_2, \ldots, P_n)$ with the affine subspace $\mathbb{R}^d \times \{\sum \frac{1}{n} e_i\}$ equals the (scaled) Minkowski sum $(\frac{1}{n} \sum P_i) \times \{\sum \frac{1}{n} e_i\}$. This way each polyhedral subdivision of the Cayley embedding $\mathrm{Cayley}(P_1, P_2, \ldots, P_n)$ induces a polyhedral subdivision of the Minkowski sum $\sum P_i$. The Cayley trick says that this correspondence gives a poset isomorphism between the polyhedral subdivisions of $\mathrm{Cayley}(P_1, \ldots, P_n)$ and the mixed subdivisions of $\sum P_i$, with regular subdivisions corresponding to coherent mixed subdivisions.

Here we have $P_1 = P_2 = \cdots = P_n = \Delta_{d-1}$, so that $\mathrm{Cayley}(\Delta_{d-1}, \ldots, \Delta_{d-1}) = \Delta_{n-1} \times \Delta_{d-1}$, and hence the Cayley trick provides a bijection between the subdivisions of the product $\Delta_{n-1} \times \Delta_{d-1}$ and mixed subdivisions of the dilated simplex $n\Delta_{d-1}$. In this case, a mixed cell τ in a mixed subdivision of $n\Delta_{d-1}$ is given by $\tau = \sum_{j=1}^n \Delta_{I_j}$ where each $\Delta_{I_j} = \mathrm{conv}\{e_i : i \in I_j\}$ is a face of $\Delta_{d=1}$ with $I_j \subseteq [d]$. Hence we can represent the cell as simply $\tau = I_1 + I_2 + \cdots + I_n$.

We have seen that (regular) subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$ are closely related to the combinatorics of tropical complexes, and the aim of this section is to interpret our results from the previous sections in terms of mixed subdivisions of dilated simplices. For this it will be convenient to have a notion of coarse type of a cell defined purely in terms of the mixed subdivision. Of course these are obtained by applying the Cayley trick to the coarse types introduced above.

Definition 4.1. Suppose $\tau = I_1 + I_2 + \cdots + I_n$ is an i-dimensional mixed cell in a mixed subdivision of $n\Delta_{d-1}$, with each $I_j \subseteq [d]$. The *coarse type* $\mathbf{t}(\tau) \in \mathbb{N}^d$ is a vector whose i-th coordinate is given by $\#\{I_j : i \in I_j\}$, the number of occurrences of i in the decomposition of τ. The *dual coarse type* $\mathbf{d}(\tau) \in \mathbb{N}^n$ is a vector whose i-th coordinate is given by $\#I_i$, the number of elements of I_i.

The following can be seen as an interpretation of Theorem 2.9 in the context of mixed subdivisions. Here we provide a complete proof here which does appeal to tropical geometry and hence applies to the more general situation of not necessarily regular subdivisions.

Proposition 4.2. *In any fine mixed subdivision Σ of $n\Delta_{d-1}$, the set of 0-dimensional cells are precisely the lattice points $n\Delta_{d-1} \cap \mathbb{Z}^d$, and the collection of coarse types of these cells are in bijection with the set of compositions of n into d parts.*

Proof. As above, we denote by $\Delta_{d-1} = \mathrm{conv}\{e_i : i \in [d]\} \subset \mathbb{R}^d$ the standard simplex. By definition, each cell τ of the subdivision Σ is of the form $\tau = \sum_{j=1}^n \Delta_{I_j}$ where $I_j \subseteq [d]$ and $\Delta_{I_j} = \mathrm{conv}\{e_i : i \in I_j\}$ is a face of Δ_{d-1}. Since we assumed the mixed subdivision Σ to be fine this yields $\dim \tau = \sum_j \dim \Delta_{I_j} = \sum_j (|I_j| - 1)$.

Any 0-dimensional cell of Σ is of the form $\{e_{i_1}\} + \cdots + \{e_{i_n}\}$, where $1 \leq i_j \leq d$. Any such cell is a lattice point in $n\Delta_{d-1} \cap \mathbb{Z}^d$, and the collection of coarse types of such cells correspond to the set of compositions of n into d parts. In order to show that all such lattice points arise as 0-dimensional cells, let $t \in n\Delta_{d-1} \cap \mathbb{Z}^d$ and let $\tau = \sum_{j=1}^n \Delta_{I_j}$ be the inclusion minimal mixed cell containing t. By [23, Prop. 14.12] every lattice point of τ is of the form $\sum_{j=1}^n \{e_{i_j}\}$ with $i_j \in I_j$ for all j. However, as the mixed subdivision is fine, τ is combinatorially isomorphic to the product $\prod_{j=1}^n \Delta_{I_j}$ and hence every sum of vertices is a vertex. Therefore, τ is a vertex of the mixed subdivision Σ. $\square$

4.2. Resolutions supported by mixed subdivisions.

In this section we discuss our results regarding cellular resolutions in the context of mixed subdivisions of dilated simplices. Although these results are more or less translations of the above via the Cayley trick, we find
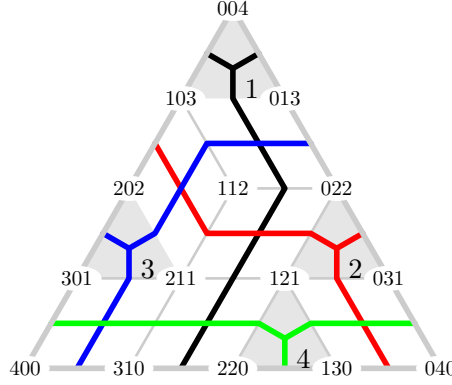
FIGURE 4. Mixed subdivision of $4\Delta_2$ corresponding to Example 2.3.

it useful to make this transition explicit. It seems that the mixed subdivision picture allows for more natural statements whereas the tropical convexity picture allows for more natural proofs. We refer the reader back to Definition 4.1 for the definition of coarse type of a mixed cell.

Corollary 4.3. *Let Σ be any regular mixed subdivision of $n\Delta_{d-1}$. Consider Σ to be a labeled polytopal complex with each face σ labeled by the least common multiple of the vertices that it contains. Then for any field $\Bbbk$, the complex $\Sigma_{\mathcal{A}}$ supports a minimal cellular resolution of the coarse type ideal $I_{\mathbf{t}(\mathcal{A})} = \langle \mathbf{x}^{\mathbf{t}(p)} : p \in \mathbb{T}^{d-1} \rangle$ in $\Bbbk[x_1, \ldots, x_d]$.*

Proof. The fact that $\Sigma_{\mathcal{A}}$ is a labeled complex follows from Proposition 3.4. As a poset $\Sigma_{\mathcal{A}}$ is isomorphic to the corresponding regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ via the Cayley trick. By [7, Lemma 22] this regular subdivision is dual to the cell decomposition $\mathcal{C}_{\mathcal{A}}$ of $\mathbb{T}^{d-1}$. In this way the labeling of $\Sigma_{\mathcal{A}}$ turns into the colabeling of $\mathcal{B}_{\mathcal{A}}$ by coarse types. Theorem 3.6 now establishes the claim. $\square$

It is now straightforward to derive the mixed subdivision result corresponding to Corollary 3.7.

Corollary 4.4. *Let Σ be any* fine *mixed subdivision of $n\Delta_{d-1}$. Then $\Sigma_{\mathcal{A}}$, as a labeled polyhedral complex, supports a minimal cellular resolution of $\langle x_1, \ldots, x_d \rangle^n$.*

Proof. This follows from Corollary 4.3 and Proposition 4.2. $\square$

We note that mixed subdivisions of dilated simplices have been used to obtain cellular resolutions in previous work. In [9] the authors study applications to resolutions of edge ideals of graphs and hypergraphs. In [29], Sinefakopoulos shows that the mixed subdivision of $n\Delta_{d-1}$ corresponding to the *staircase* triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ supports a cellular resolution of $\langle x_1, \ldots, x_d \rangle^n$, although his construction is much less explicit. We discuss this example further in Section 5.

Remark 4.5. It seems to be a challenging task to characterize which monomial ideals $I \subset \Bbbk[x_1, \ldots, x_d]$ arise as $I_{\mathbf{t}(\mathcal{A})}$ for some arrangement $\mathcal{A}$ of tropical hyperplanes in $\mathbb{T}^{d-1}$. Some necessary conditions are obvious, e.g., I should be homogeneous of some degree n, and that x_i^d should be contained in I for all i. In particular, this means that the coarse type ideal is necessarily Artinian.

4.3. **Alexander duality of ideals and resolutions.** We have seen how the bounded sub-complexes of tropical hyperplane arrangements are related to mixed subdivisions of dilated simplices in terms of a geometric duality. This duality extends to the algebraic level of our resolutions in the context of *Alexander duality of resolutions.* For this we will need the following notion of the Alexander dual of a (not necessarily square-free) monomial ideal.

Definition 4.6. Suppose I is a monomial ideal in the polynomial ring $\Bbbk[x_1, \ldots, x_d]$ and let $\mathbf{a} \in \mathbb{N}^d$. The *Alexander dual of I with respect to* $\mathbf{a}$ is given by the intersection

$$I^{[\mathbf{a}]} \ = \ \bigcap \left\{ \mathfrak{m}^{\mathbf{a} \backslash \mathbf{b}} \ : \ \mathbf{x}^{\mathbf{b}} \text{ is a minimal generator of } I \right\},$$

where $\mathbf{a} \backslash \mathbf{b}$ denotes the vector whose i-th coordinate is $a_i + 1 - b_i$ if $b_i \geq 1$, and is 0 if $b_i = 0$. Here we borrow the notation $\mathfrak{m}^{\mathbf{a}} := \langle x_i^{a_i} : a_i \geq 1 \rangle$.

Note that if I is a square-free monomial ideal (and hence the Stanley-Reisner ring of some simplicial complex) and $\mathbf{a} = \mathbb{1}$ is taken to be the all-ones vector, then this notion recovers the more familiar notion of Alexander duality of simplicial complexes. The main result concerning duality of resolutions, relevant for us, is the following ([21, Theorem 5.37]).

Theorem 4.7. *Suppose I is a monomial ideal in degrees preceding some $\mathbf{a} \in \mathbb{N}^d$ and suppose $\mathcal{F}_\bullet^{\mathcal{P}}$ is a minimal cellular resolution of $S/(I + \mathfrak{m}^{\mathbf{a}+\mathbb{1}})$ such that all face labels on $\mathcal{P}$ precede $\mathbf{a} + \mathbb{1}$. Let $\mathcal{Q}$ denote the labeled complex with the same underlying complex $\mathcal{P}$ but with labels $\overline{\mathbf{t}}_F = \mathbf{a} + \mathbb{1} - \mathbf{t}_F$. Then $\mathcal{F}_\bullet^{\mathcal{Q}_{\leq \mathbf{a}}}$ is a minimal cocellular resolution of $I^{[\mathbf{a}]}$.*

Applying this theorem we obtain the following dual resolution of the coarse cotype ideal. For σ a face of a mixed subdivision Σ, the *coarse cotype* of σ is defined to be $n\mathbb{1} - \mathbf{t}(\sigma)$, where $\mathbf{t}(\sigma)$ is the coarse type of σ defined in Definition 4.1.

Proposition 4.8. *Given any arrangement $\mathcal{A}$ of n tropical hyperplanes in $\mathbb{T}^{d-1}$, let $\overline{\Sigma}_{\mathcal{A}}$ denote the associated mixed subdivision of $n\Delta_{d-1}$ with labels given by coarse cotype. Then $\overline{\Sigma}_{\mathcal{A}}$ supports a minimal cocellular resolution of the coarse cotype ideal $I_{\overline{\mathbf{t}}(\mathcal{A})}$ in $\Bbbk[x_1, \ldots, x_d]$. Consequently the associated tropical complex $\mathcal{B}_{\mathcal{A}}$, with labels given by coarse cotype, supports a minimal cellular resolution of $I_{\overline{\mathbf{t}}(\mathcal{A})}$.*

Proof. We apply Theorem 4.7 with $\mathbf{a} := (n-1)\mathbb{1}$, and with I defined to be the ideal generated by all monomials f of $I_{\mathbf{t}(\mathcal{A})}$ with $f \leq \mathbf{a}$ (in other words, throw out all generators of the form x_i^n for $1 \leq i \leq d$, all of which show up in $I_{\mathbf{t}(\mathcal{A})}$ regardless of the arrangement $\mathcal{A}$). We then have from Corollary 4.3 that $\Sigma_{\mathcal{A}}$ supports a minimal cellular resolution of $I_{\mathbf{t}(\mathcal{A})} = I + \mathfrak{m}^{\mathbf{a}+\mathbb{1}}$. The conditions of Theorem 4.7 are met and we conclude that $(\overline{\Sigma}_{\mathcal{A}})_{\leq \mathbf{a}}$ supports a minimal cocellular resolution of $I^{[\mathbf{a}]}$.

We next determine $I^{[\mathbf{a}]}$, the Alexander dual of I with respect to $\mathbf{a} = (n-1)\mathbb{1}$. In [21] it is shown that if $\mathbf{b} \leq \mathbf{a}$ then $\mathbf{x}^{\mathbf{b}}$ lies outside I if and only if $\mathbf{x}^{\mathbf{a}-\mathbf{b}}$ lies inside $I^{[\mathbf{a}]}$. Hence to find a set of generators for $I^{[\mathbf{a}]}$ it suffices to determine the maximal monomials which lie outside I. But these monomials correspond to the minimal cotypes that arise in the complex $\mathcal{C}_{\mathcal{A}}$, and these are given by the collection of monomials $\overline{\mathbf{t}}(x)$, for x a 0-dimensional cell in $\mathcal{C}_{\mathcal{A}}$. Hence we conclude that $I^{[\mathbf{a}]} = I_{\overline{\mathbf{t}}(\mathcal{A})}$.

For the second part of the claim, we note that a face $\sigma \in \overline{\Sigma}_{\mathcal{A}}$ with label $\mathbf{b}$ satisfies $\mathbf{b} \leq \mathbf{a}$ if and only if $n\mathbb{1} - \mathbf{t}(\sigma) \leq (n-1)\mathbb{1}$ for the coarse type label $\mathbf{t}(\sigma)$ in $\Sigma_{\mathcal{A}}$. But this occurs exactly when $\mathbf{t}(\sigma) \geq \mathbb{1}$, which happens if and only if the face σ is not contained in the boundary of $\Sigma_{\mathcal{A}}$. Hence the cocellular resolution of $I^{[\mathbf{a}]}$ that we obtain, supported on $(\overline{\Sigma}_{\mathcal{A}})_{\leq \mathbf{a}}$, is given by the relative cocellular complex of $(\Sigma_{\mathcal{A}}, \partial \Sigma_{\mathcal{A}})$. By duality, the relative cochain complex

$C^*(\Sigma_{\mathcal{A}}, \partial \Sigma_{\mathcal{A}})$ is isomorphic to the chain complex $C_*(\mathcal{B}_{\mathcal{A}})$ of the tropical complex $\mathcal{B}_{\mathcal{A}}$ (the bounded subcomplex of the decomposition of $\mathbb{T}^{d-1}$ induced by $\mathcal{A}$). Hence $\mathcal{B}_{\mathcal{A}}$, with labels given by coarse *cotype*, supports a minimal *cellular* resolution of the ideal $I^{[\mathbf{a}]} = I_{\bar{\mathfrak{t}}(\mathcal{A})}$. $\square$

5. Examples

In this section we discuss some examples of our constructions and results. We begin with a family of generic arrangements which correspond to well-known objects in the theory of subdivisions (staircase triangulations) and tropical convexity (cyclic polytopes). In the context of the associated coarse type ideal we recover a construction of a cellular resolution first described by Sinefakopoulos in [29]. We then discuss a family of non-generic examples which arise from tropical hypersimplices; in this we are able to explicitly describe the generators of the coarse type ideal.

5.1. **The staircase triangulation.** One particularly well-behaved fine mixed subdivision of $n\Delta_{d-1}$ comes from applying the Cayley trick (discussed in Section 4.1 above) to the so-called *staircase* triangulation of $\Delta_{n-1} \times \Delta_{d-1}$. The origins of the staircase triangulation date back at least to work of Eilenberg and Zilber in [10], where the authors introduce algebraic operations which they call 'shuffles' to study the homology groups of products of spaces. We wish to recall the construction here in language suitable for our purposes.

As above, we use $\{e_1, \ldots, e_d\}$ to denote the vertices of the simplex Δ_{d-1} and consider fine mixed cells of the following kind. Given a sequence of integers $(b_1, b_2, \ldots, b_{n+1})$ satisfying $1 = b_1 \leq b_2 \leq \cdots \leq b_n \leq b_{n+1} = d$ we let $B_i := \{e_{b_i}, e_{b_i+1}, \ldots, e_{b_{i+1}}\}$ for $1 \leq i \leq n$, and use $(b_1, b_2, \ldots, b_{n+1})$ to denote the corresponding (fine) mixed cell $B_1 + B_2 + \cdots + B_n$ of $n\Delta_{d-1}$. Note that each such sequence can be thought of as a 'staircase' of length n and height d, where at the i-th step one should climb a step of height $b_{i+1} - b_i$.

We claim that the collection of fine mixed cells corresponding to $(b_1, b_2, \ldots, b_{n+1})$ for $1 = b_1 \leq b_2 \leq \cdots \leq b_n \leq b_{n+1} = d$ forms a fine mixed subdivision of the complex $n\Delta_{d-1}$. To see this, we once again employ the Cayley trick and consider the (proposed) triangulation of the product $\Delta_{n-1} \times \Delta_{d-1}$. As discussed above, the vertices of $\Delta_{n-1} \times \Delta_{d-1}$ are pairs (e_i, e_j), where $1 \leq i \leq n$ and $1 \leq j \leq d$. A simplex in any triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ corresponds to a tree in the complete bipartite graph $K_{n,d}$, where (v_i, w_j) is an edge of the graph whenever (e_i, e_j) is a vertex of the simplex. The corresponding mixed cell $B_1 + \cdots + B_n$ of $n\Delta_{d-1}$ has as the i-th summand the face $\{e_j : w_j \in N(v_i)\}$, given by all vertices that are adjacent to v_i. Note that the coarse type of the mixed cell is given by the degree sequence of the d vertices in the second vertex set partition.

Working with $\Delta_{n-1} \times \Delta_{d-1}$ has the advantage that one can readily verify if a collection of simplices does, in fact, give a triangulation of the space, in terms of the corresponding subgraphs. We omit the details here, but point out that the staircase triangulation also arises from the so-called 'cyclic arrangement' of n tropical hyperplanes in $\mathbb{T}^{d-1}$ as discussed in [4]. We call the associated fine mixed subdivision of $n\Delta_{d-1}$ the *staircase mixed subdivision*. As with any fine mixed subdivision, the 0-dimensional cells of the staircase mixed subdivision of $n\Delta_{n-1}$ are labeled by the monomial generators of $\langle x_1, \ldots, x_d \rangle^n$, and from Corollary 4.4 we know that this labeled complex supports a minimal cellular resolution of $\langle x_1, \ldots, x_d \rangle^n$. In fact, the case of $n = d = 3$ is hinted at in [21, Example 2.20].

Remark 5.1. In [29], Sinefakopoulos constructs a labeled polyhedral complex which he calls $P_n(x_1, \ldots, x_d)$, and shows that it supports the minimal resolution of $\langle x_1, \ldots, x_d \rangle^n$. The complex $P_n(x_1, \ldots, x_d)$ is constructed inductively, with each $P_n(x_{k+1}, \ldots, x_d)$ a subcomplex

of $P_n(x_k, \ldots, x_d)$ for all $k < d$. In [29] it is shown that $P_n(x_1, \ldots, x_d)$ can in fact be realized as a subdivision of the dilated simplex $n\Delta_{d-1}$, and here we claim that this complex is isomorphic (as a labeled complex) to the staircase mixed subdivision.

Mimicking the construction of $P_n(x_1, \ldots, x_d)$ from [29], we proceed by induction on n. If $n = 1$, both $P_1(x_1, \ldots, x_d)$ and the staircase mixed subdivision correspond to the standard $(d-1)$-dimensional simplex with vertex labels $\{x_1, \ldots, x_d\}$, denoted $\Delta_{d-1}(x_1, \ldots, x_d)$. We assume that for $n \geq 1$, the complex $P_n(x_1, \ldots, x_d)$ is isomorphic as a labeled complex to the staircase subdivision of $n\Delta_{d-1}$, with this isomorphism restricting to an isomorphism on each $P_n(x_k, \ldots, x_d)$. In [29] the inductive step of the construction is obtained as

$$P_{n+1}(x_1, \ldots, x_d) \; := \; C_1 \cup \cdots \cup C_d \,,$$

where each C_k is defined as

$$C_k \; := \; \Delta_{k-1}(x_1, \ldots, x_k) \times P_n(x_k, \ldots x_d) \,.$$

For any k, we claim that the complex C_k corresponds to the complex composed of the set of fine mixed cells $(b_1, b_2, \ldots, b_{n+1})$ with $b_2 = k$. This would establish the desired isomorphism since the set of fine mixed cells in the staircase mixed subdivision correspond to the all such sequences with $1 \leq b_2 \leq d$. To prove the claim, we note that for any labeled complex P the product $\Delta_{k-1}(x_1, \ldots, x_k) \times P$ is isomorphic to $\Delta_{k-1}(x_1, \ldots, x_k) + P$, assuming that these complexes lie in affinely independent subspaces. Furthermore, the vertices of the product complex are labeled by sums of vertices from each summand. Now, by induction we have that each $P_n(x_k, \ldots, x_d)$ is isomorphic to the corresponding subcomplex of the staircase mixed subdivision of $n\Delta_{d-1}$. In particular, the set of mixed cells correspond to the set of sequences $(b_1, b_2, \ldots, b_{n+1})$ with $k = b_1 \leq b_2 \cdots \leq b_n \leq b_{n+1} = d$. The facets of C_k are obtained by taking the Minkowski sum of the simplex $\Delta_{k-1}(x_1, \ldots, x_k)$ with these mixed cells, and hence C_k is isomorphic to the complex composed of all fine mixed cells $(b_1, b_2, \ldots, b_{n+1})$ with $b_2 = k$.

Hence our constructions recover the result from [29] (with a more explicit description of the underlying polyhedral complex). Once again, we wish to emphasize that from Corollary 4.4 we know that in fact *any* (regular) fine mixed subdivision of $n\Delta_{d-1}$ supports a minimal cellular resolution of the homogeneous ideal $\langle x_1, \ldots, x_d \rangle^n$.

5.2. **Tropical hypersimplices.** The hypersimplex $\Delta(k, n)$ is an ordinary convex polytope which, e.g., naturally turns up in the study of tropical Grassmannians of tropical k-planes in $\mathbb{T}^{n-1}$. Its tropical counterpart, the *tropical hypersimplex* $\Delta^{\mathrm{trop}}(k, n)$ is defined as the tropical convex hull of all $0/1$-vectors of length n with exactly k zeros. Notice that we have the strict inclusions

$$\Delta^{\mathrm{trop}}(1, n) \; \supsetneq \; \Delta^{\mathrm{trop}}(2, n) \; \supsetneq \; \cdots \; \supsetneq \Delta^{\mathrm{trop}}(n-1, n)$$

of subsets of $\mathbb{T}^{n-1}$. We want to determine the coarse type ideal corresponding to the configuration of $\binom{n}{k}$ points in $\mathbb{T}^{n-1}$ given by the tropical vertices of $\Delta^{\mathrm{trop}}(k, n)$. The n generators of $\Delta^{\mathrm{trop}}(1, n)$ are in general position. Hence the coarse type ideal is the homogeneous maximal ideal $\langle x_1, x_2, \ldots, x_n \rangle$ in this case. The second tropical hypersimplex $\Delta^{\mathrm{trop}}(2, n)$ is contained in the min-tropical hyperplane with the origin as its apex. In particular, $\Delta^{\mathrm{trop}}(2, n)$, seen as a polytopal complex in $\mathbb{R}^{n-1} = \mathbb{T}^{n-1}$, is of dimension $n - 2$. This implies that all maximal cells in the type decomposition of $\mathbb{T}^{n-1}$ induced by the tropical vertices of $\Delta^{\mathrm{trop}}(k, n)$ are unbounded if $2 \leq k < n$. Equivalently, the coarse type ideal only has minimal generators with the property that at least one variable is missing (that is, the exponent of this variable is zero). The symmetric group $\mathrm{Sym}(n)$ acts on the set of tropical vertices of $\Delta^{\mathrm{trop}}(k, n)$, and hence it also acts on the maximal cells of the type decomposition.

Proposition 5.2. *Let $2 \leq k < n$. Then up to the $\mathrm{Sym}(n)$ action the coarse types of the maximal cells in the type decomposition of $\mathbb{T}^{n-1}$ induced by the $\binom{n}{k}$ tropical vertices of $\Delta^{\mathrm{trop}}(k, n)$ are given by*

$$\left(\binom{n-\alpha}{k} + \binom{n-1}{k-1}, \ \binom{n-2}{k-1}, \ldots, \binom{n-\alpha}{k-1}, \ \underbrace{0, \ldots, 0}_{n-\alpha} \right)$$

where $1 \leq \alpha \leq n - k + 1$.

This result has been obtained by Katja Kulas, and a complete proof will appear in [15]. Here we only sketch the argument. Let $\mathcal{A}$ denote the arrangement of max-tropical hyperplanes induced by the tropical vertices of $\Delta^{\mathrm{trop}}(k, n)$. The maximal dimension of a bounded cell or, equivalently, the tropical rank of the matrix whose columns are the tropical vertices of $\Delta^{\mathrm{trop}}(k, n)$, equals $n - k$ [6, Proposition 7.2]. All cells in the arrangement $\mathcal{A}$, bounded or not, are convex polyhedra in $\mathbb{R}^{\binom{n}{k}-1}$ which are *pointed*, that is, they do not contain any affine line. This implies that each cell must contain a bounded cell as a face. Let C be some maximal cell in the arrangement $\mathcal{A}$. From now on we assume that $k > 1$ whence C is necessarily unbounded. Let $\alpha - 1$ be the maximal dimension of a bounded cell in the boundary of C; and we call α the *class* of C. From the bound on the tropical rank it follows that $1 \leq \alpha \leq n - k + 1$. All these values for α actually occur. It turns out that $\mathrm{Sym}(n)$ acts transitively on the set of maximal cells of class α. The coarse type of one representative of each class is given in Proposition 5.2. Due to Corollary 3.5 this yields generators for the corresponding coarse type ideal.

6. Face counting and incidence structure of the tropical complex

In Section 3 we saw how the polyhedral complexes $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{A}}$ associated to an arrangement $\mathcal{A}$ gave rise to resolutions of the coarse type ideal $I_{\mathbf{t}(\mathcal{A})}$ and the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$, respectively. The *minimality* of our resolution also leads to some important implications regarding the combinatorics of $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{A}}$ themselves. In this section we discuss face numbers of tropical complexes, as well as an algorithm for determining the facial structure of $\mathcal{B}_{\mathcal{A}}$ given the arrangement $\mathcal{A} = \mathcal{A}(V)$. The latter generalizes a result of [4], where a similar algorithm for the case of sufficiently generic arrangements was provided.

6.1. **Counting faces.** As a first application we point out that the f-vector of $\mathcal{C}_{\mathcal{A}}$ can be determined from the $\mathbb{Z}$-graded ('coarse') Betti numbers of $I_{\mathbf{t}(\mathcal{A})}$. We noted in Proposition 2.12 that from the coarse type it is possible to distinguish bounded from unbounded cells in $\mathcal{C}_{\mathcal{A}}$. Thus, we can also recover the numerical behavior of the bounded complex $\mathcal{B}_{\mathcal{A}}$.

Corollary 6.1. *Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and let $I_{\mathbf{t}(\mathcal{A})}$ be its coarse type ideal. Then the number of cells in $\mathcal{C}_{\mathcal{A}}$ of dimension k is*

$$f_k(\mathcal{C}_{\mathcal{A}}) = \beta_{d-1-k}(I_{\mathbf{t}(\mathcal{A})}) = \sum_{\mathbf{b} \in \mathbb{Z}^d} \beta_{d-1-k, \mathbf{b}}(I_{\mathbf{t}(\mathcal{A})}).$$

The number of bounded k-cells in $\mathcal{C}_{\mathcal{A}}$ is given by as the sum of Betti numbers $\beta_{d-1-k, \mathbf{b}}(I_{\mathbf{t}(\mathcal{A})})$ for which $\mathbf{b} > 0$.

Note that for the k-cells we need to consider the $(d-1-k)$-th Betti numbers. This is due to the fact that $\mathcal{C}_{\mathcal{A}}$ supports a *co*cellular resolution.

Furthermore, we can use the *uniqueness* of minimal resolutions to derive further results regarding face numbers of arrangements. For this, suppose $\mathcal{A}$ is a sufficiently generic arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$, and let $\mathcal{C}_\mathcal{A}$ denote the induced polyhedral subdivision of $\mathbb{T}^{d-1}$ determined by type. In Corollary 3.7 we saw that $\mathcal{C}_\mathcal{A}$, with labels given by coarse type, supports a minimal cocellular resolution of the ideal $\langle x_1, \ldots, x_d \rangle^n$. Any two such resolutions are isomorphic as chain complexes, and in particular the finely graded Betti numbers $\beta_{i,\sigma}$ do not depend on the resolution. By construction, $\beta_{i,\sigma}$ is precisely the number of cells in $\mathcal{C}_\mathcal{A}$ with monomial label σ. But the monomial labels are given by the coarse types, and hence we obtain the following.

Corollary 6.2. *Let $\mathcal{A}$ be a sufficiently generic arrangement of n hyperplanes in $\mathbb{T}^{d-1}$. For every $0 \leq i \leq d-1$ the collection of coarse types $\mathbf{t}(C_T)$ for $\dim C_T = i$, counted with multiplicities, is independent of the arrangement.*

Putting the above result in perspective with the second statement of Corollary 6.1, this proves the following result without appealing to the equidecomposability of the product of simplices.

Corollary 6.3. *The number of cells of $\mathcal{C}_\mathcal{A}$ for a tropical hyperplane arrangement $\mathcal{A}$ in general position is independent of the choice of hyperplanes. More precisely, the number of k-dimensional cells induced by the arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$ equals*

$$f_k(\mathcal{C}_A) = \sum_{\ell=0}^{k} \binom{n+d-2-\ell}{n-1}\binom{d-1-\ell}{d-1-k}.$$

Proof. By Corollaries 6.1 and 3.7 we have $f_i(\mathcal{C}_\mathcal{A}) = \beta_{d-1-i}(\langle x_1, \ldots, x_n\rangle^n)$. The Betti numbers of the power of the homogeneous maximal ideal are well-known. For example, they can be determined as follows.

The ideal $\mathfrak{m}^n = \langle x_1, \ldots, x_d \rangle^n$ is *strongly stable*, that is, $\frac{x_i}{x_j}\mathbf{x}^\mathbf{b} \in \mathfrak{m}^n$ for every $\mathbf{x}^\mathbf{b}$ monomial divisible by x_j and $i < j$. In particular, $\mathfrak{m}^n$ is Borel-fixed, and hence the Betti numbers are given by [21, Thm. 2.18], and we have

$$\beta_i(\mathfrak{m}^n) = \sum_{\mathbf{a}\in\mathbb{N}^d,|a|=n} \binom{\max(\mathbf{x}^\mathbf{a})-1}{i}$$

where $\max(\mathbf{x}^\mathbf{a}) = \max\{i : \mathbf{a}_i > 0\}$. Now if $\max(\mathbf{x}^\mathbf{a}) = \ell$, then

$$\mathbf{a} = (a_1, a_2, \ldots, a_\ell + 1, 0, \ldots, 0)$$

with $a_1, \ldots, a_\ell \geq 0$ and $\sum_i a_i = n-1$. Hence, the number of generators $\mathbf{x}^\mathbf{a}$ with $\max(\mathbf{x}^a) = \ell$ is the number of monomials in ℓ variables of total degree $n-1$. This yields

$$\beta_i(\mathfrak{m}^n) = \sum_{\ell=1}^{d} \binom{n-2+\ell}{n-1}\binom{\ell-1}{i}$$

Now substitute i in the formula with $d-1-k$. $\qquad\qquad\square$

In the context of mixed subdivisions of dilated simplices, the previous two corollaries yield the following.

Corollary 6.4. *In any regular fine mixed subdivision Σ of $n\Delta_{d-1}$ the collection of coarse types, counted with multiplicities, is independent of the subdivision. In particular, $f_i(\Sigma)$, the*

number of i-dimensional faces of Σ is independent of the subdivision, and is given by

$$f_i(\Sigma) = \sum_{\ell=1}^{d} \binom{n-2+\ell}{n-1} \binom{\ell-1}{i}.$$

Remark 6.5. For the case of vertices, Corollary 6.4 was already established in 4.2, where the it was shown that the 0-dimensional cells in any fine mixed subdivision correspond to the lattice points of $n\Delta_{d-1}$. The case of facets can also be proven directly using the following combination of mixed volume calculations and the switch duality of tropical complexes. Given a fine mixed subdivision Σ of $n\Delta_{d-1}$ we consider the corresponding fine mixed subdivision Σ^* of $d\Delta_{n-1}$ induced by applying the Cayley trick and switching the factors in $\Delta_{n-1} \times \Delta_{d-1}$. For nonnegative real numbers $\lambda_1, \ldots, \lambda_d$, we have that the linear functional given by the volume $\mathrm{Vol}(\lambda_1\Delta_{n-1} + \cdots + \lambda_d\Delta_{n-1})$ can be expressed as a homogeneous polynomial of degree $n-1$ in the variables $\lambda_1, \ldots, \lambda_d$. The coefficient of the monomial $\lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_d^{i_d}$ is called a mixed volume, and for any fine mixed subdivision of $d\Delta_{n-1}$ it is equal to the sum of the volumes of the mixed cells of the corresponding *dual* coarse type. In our case all fine mixed cells of a certain dual coarse type have the same volume, since the relevant data is given by the number of vertices in each factor, regardless of the labels on the vertices. This implies that in any fine mixed subdivision the number of fine mixed cells with a certain dual coarse type is always the same. But the collection of *dual coarse* types for this fine mixed subdivision coincides with collection of *coarse* types for the given fine mixed subdivision of $n\Delta_{d-1}$, proving our claim.

In fact (for $k = 0$), each coarse type shows up at most once and the collection of coarse types can be seen to coincide with the monomial generators of the ideal

$$x_1 x_2 \cdots x_d \langle x_1, x_2, \ldots, x_d \rangle^{n-1}.$$

This follows from the fact that in this case the volume polynomial of the dual mixed subdivision is given by $(\lambda_1 + \cdots + \lambda_d)^{n-1}$ and the volume of a mixed cell of a particular dual coarse type is exactly the coefficient of the corresponding monomial. We do not know of a similar mixed volume interpretation of the other face numbers.

6.2. **Incidence structure of the bounded complex from the fine cotype ideal.** In [4] Block and Yu develop an algorithm to compute the bounded complex of a generic tropical hyperplane arrangement $\mathcal{A} = \mathcal{A}(V)$ that draws from computational commutative algebra. There it is shown that the bounded complex supports a minimal cellular resolution of a monomial ideal which we have called the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$ associated to $\mathcal{A}$. It turns out that this ideal is an initial ideal of the toric ideal $I_{n,d}$ for the vertices of $\Delta_{n-1} \times \Delta_{d-1}$. The results in [4] rely on the genericity of the arrangement in two ways: i) the fact that the bounded complex supports a resolution of the fine cotype ideal is proved by appealing to the *polyhedral detour* to tropical convexity presented in [7], and ii) the genericity is needed to guarantee that the initial ideal is indeed monomial.

In this section we extend their algorithm to the case of non-generic hyperplane arrangements. We bypass the two mentioned dependences on genericity as follows. In Section 3 we have already shown that the bounded complex resolves the fine cotype ideal for an *arbitrary* arrangement, using first principles in tropical convexity. As for ii), we next show that the fine cotype ideal is the maximal monomial ideal contained in the initial ideal associated to the weights V. In terms of polyhedral geometry, this corresponds to the passage from the polyhedral subdivision Σ_V of $\Delta_{n-1} \times \Delta_{d-1}$ induced by V to the *crosscut complex*.

The results of the previous sections cannot directly be turned into a practical algorithm as the fine cotype ideal can only be determined *after* computing the bounded complex or, at least, the vertices of $\mathcal{C}_\mathcal{A}$. In the case of a generic arrangement of tropical hyperplanes this problem is resolved in [4] as follows. The ideal

$$I_{n,d} \ = \ \langle x_{ik}x_{jl} - x_{il}x_{jk} : i, j \in [n], k, l \in [d] \rangle$$

of 2×2-minors of a general $n \times d$-matrix in $\Bbbk[x_{11}, \ldots, x_{nd}]$ is the toric ideal associated to the vertices of the ordinary lattice polytope $\Delta_{n-1} \times \Delta_{d-1}$. The initial ideal $\mathrm{in}_V(I_{n,d})$ with respect to the weights V is a squarefree monomial ideal and, by a celebrated result of Sturmfels [31, Thm. 8.3], equal to the Stanley-Reisner ideal I_Σ of the triangulation Σ of $\Delta_{n-1} \times \Delta_{d-1}$ induced by V; see Section 4. The face poset of the bounded complex $\mathcal{B}_\mathcal{A}$ is anti-isomorphic to the subposet of *interior* cells of Σ. In short, there is a bijection between i-cells of $\mathcal{C}_\mathcal{A}$ and $(n + d - 2 - i)$-cells of Σ which lie in the interior of $\Delta_{n-1} \times \Delta_{d-1}$. The bijection takes a cell C_T° of fine type T to the cell of Σ with vertices (e_i, e_j) for $T_{ij} = 1$. Finally, the Alexander dual of I_Σ is the monomial ideal generated by the fine cotypes of the vertices of $\mathcal{C}_\mathcal{A}$.

If the placement V of the hyperplanes is not generic, then the induced subdivision Σ is not a triangulation. However, the above bijection is unaffected and, in particular, the fine type of a vertex of $\mathcal{C}_\mathcal{A}$ can be read off the corresponding facet of Σ. We define the *crosscut complex* $\mathrm{CrossCut}(\Sigma) \subseteq 2^{[n] \times [d]}$ to be the unique simplicial complex with the same vertices-in-facets incidences as the polyhedral complex Σ. The crosscut complex is a standard notion in combinatorial topology and can be defined in more generality (see Björner [3, pp. 1850ff]). The following observation is immediate from the definitions.

Proposition 6.6. *For V an ordered sequence of n points in $\mathbb{T}^{d-1}$ let $\mathcal{A} = \mathcal{A}(V)$ be the corresponding tropical arrangement and let Σ be the regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ induced by V. Then the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$ is Alexander dual to the Stanley-Reisner ideal of $\mathrm{CrossCut}(\Sigma)$.*

The crosscut complex encodes the information of which collections of vertices lie in a common face. Hence, the crosscut complex is a purely combinatorial object and does not see the affine structure of the underlying polyhedral complex.

Algebraically, the initial ideal $\mathrm{in}_V(I_{n,d})$ is a coherent A-graded ideal in the sense of [31, Ch. 10] and encodes the corresponding polyhedral subdivision Σ induced by V (see [31, Thm. 10.10]). The ideal $\mathrm{in}_V(I)$ of a toric ideal I is generated by monomials and binomials. Intuitively, the binomials encode the affine structure within cells of Σ, that is, the affine dependencies, while the monomial generators encode the Stanley-Reisner data for the crosscut complex. For an arbitrary ideal J, denote by $M(J)$ the largest monomial ideal contained in J.

Lemma 6.7. *Let I_A be the toric ideal for $A \in \mathbb{N}^{d \times n}$ and for $\omega \in \mathbb{R}^n$ let $J = \mathrm{in}_\omega(I_A)$ and Σ the regular subdivision of A induced by ω. Then the radical of $M(J)$ is the Stanley-Reisner ideal of the crosscut complex of Σ_ω, that is*

$$I_{\mathrm{Rad}(M(J))} \ = \ I_{\mathrm{CrossCut}(\Sigma)} \, .$$

Proof. By Theorem 10.10 of [31], the ideal J is the intersection of ideals J_σ indexed by the faces $\sigma \in \Sigma$ and J_σ is torus isomorphic to $I_\sigma = I_A + \langle x_i : i \notin \sigma \rangle$. Hence, we have

$$M(J) \ = \ \bigcap_{\sigma \in \Sigma} M(I_\sigma).$$

Under the projection map $\Bbbk[x_1, \ldots, x_n] \to \Bbbk[x_i : i \in \sigma]$ that takes $x_i \mapsto 0$ for $i \notin \sigma$, I_σ is isomorphic to the toric ideal corresponding to the columns of A indexed by $\sigma \subseteq [n]$. Hence, a monomial $\mathbf{x^a}$ is contained in I_σ if and only if $\operatorname{supp}(\mathbf{x}^a) \not\subseteq \sigma$. Therefore $M(I_\sigma) = \langle x_i : i \notin \sigma \rangle$ and for $\tau \subseteq [n]$ we have that $\mathbf{x}^\tau \in \operatorname{Rad}(M(J))$ if and only if τ is not contained in any cell of Σ. $\qquad\square$

A special case of this lemma appears in [25, Lem. 4.5.4]. Since the toric ideal $I_{n,d}$ is unimodular, the ideal $M(\operatorname{in}_V(I_{n,d}))$ is automatically squarefree and we obtain the following corollary.

Corollary 6.8. *Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement and let $J = \operatorname{in}_V(I_{n,d})$ the initial ideal of $I_{n,d}$ for the weights V. Then the Alexander dual of the squarefree monomial ideal $M(J)$ is the fine cotype ideal of $\mathcal{A}$.*

In the case that $\mathcal{A}$ is generic, the ideal $M(J)$ coincides with the initial ideal $\operatorname{in}_V(I_{n,d})$ and hence recovers the main result of [4]. Moreover, it entails the modification of the algorithm in [4] by replacing the ideal $\operatorname{in}_V(I_{n,d})$. We describe our Algorithm A below. For an overview of other algorithms to produce the same output via techniques from polyhedral combinatorics and algorithmic geometry see [13].

---

input : matrix $V \in \mathbb{R}^{n \times d}$
output: face poset of $\mathcal{B}_{\mathcal{A}(V)}$
1 calculate the initial ideal $J = \operatorname{in}_V(I_{n,d})$
2 calculate $M(J) = \langle \mathbf{x^a} : \mathbf{x^a} \in J \rangle$
3 calculate the Alexander dual $M(J)^*$
4 find a minimal fine graded resolution to determine the face poset of $\mathcal{B}_{\mathcal{A}(V)}$

Algorithm A: Computing the face poset of the tropical complex

An algorithm for calculating $M(\cdot)$ for a toric ideal is given in [25, Algo. 4.4.2] in terms of elimination theory. Computing the Alexander dual of a finite simplicial complex can be accomplished via an algorithm of Lawler [17]. In [16] La Scala and Stillman give an overview of methods to compute minimal resolutions. To illustrate the results in this section, we show an example computed with the computer algebra system `Macaulay2` [12].

Example 6.9. Consider the points $v_1 = (0, 1, 1)$, $v_2 = (0, 0, 1)$, and $v_3 = (0, 1, 0)$ in $\mathbb{T}^2$. The induced type decomposition is shown in Figure 5. We have

$$
\begin{aligned}
I_{3,3} \;=\; \langle & x_{11}x_{22} - \underline{x_{12}x_{21}}, \; x_{11}x_{33} - \underline{x_{13}x_{31}}, \; \underline{x_{12}x_{31}} - x_{11}x_{32}, \\
& x_{12}x_{33} - \underline{x_{13}x_{32}}, \; \underline{x_{13}x_{21}} - x_{11}x_{23}, \; x_{13}x_{22} - \underline{x_{12}x_{23}}, \\
& x_{21}x_{33} - \underline{x_{23}x_{31}}, \; x_{22}x_{31} - \underline{x_{21}x_{32}}, \; x_{22}x_{33} - \underline{x_{23}x_{32}} \rangle,
\end{aligned}
$$

where the initial forms of the generators with respect to the weight matrix $V = \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix} \right) \in \mathbb{R}^{3 \times 3}$ are underlined; these generate the toric ideal J. Its maximal monomial subideal equals

$$
M(J) \;=\; \langle x_{12}x_{21}, \; x_{12}x_{23}, \; x_{13}x_{31}, \; x_{23}x_{31}, \; x_{13}x_{32}, \; x_{21}x_{32}, \; x_{23}x_{32} \rangle,
$$

and this is squarefree. Its Alexander dual is

$$
M(J)^* \;=\; \langle x_{13}x_{21}x_{23}, \; x_{12}x_{13}x_{23}x_{32}, \; x_{12}x_{31}x_{32}, \; x_{21}x_{23}x_{31}x_{32} \rangle.
$$
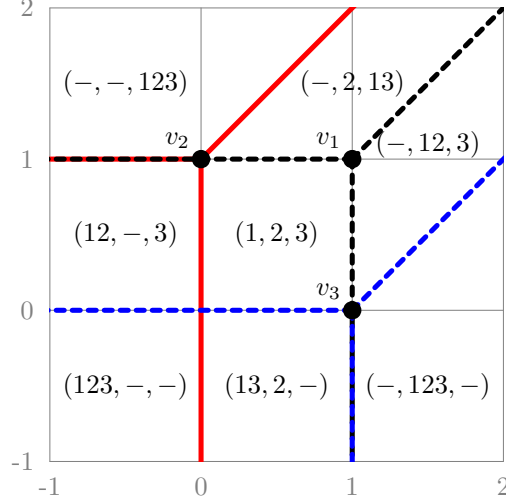
FIGURE 5. Non-generic arrangement in $\mathbb{T}^2$ with labeling by fine type.

These four generators of $M(J)^*$ encode the fine cotypes of the points v_3, $(0,0,0)$, v_2, and v_1, respectively. Setting $S = \Bbbk[x_{11}, x_{12}, \ldots, x_{33}]$ we obtain the minimal free resolution

$$0 \to S \xrightarrow{\phi_3} S^4 \xrightarrow{\phi_2} S^4 \xrightarrow{\phi_1} I \to 0,$$

where the non-trivial differentials ϕ_i are given by the matrices

$$\phi_1 = \begin{pmatrix} x_{13}x_{21}x_{23} & x_{12}x_{31}x_{32} & x_{21}x_{23}x_{31}x_{32} & x_{12}x_{13}x_{23}x_{32} \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} 0 & -x_{31}x_{32} & -x_{12}x_{32} & 0 \\ -x_{21}x_{23} & 0 & 0 & -x_{13}x_{23} \\ x_{12} & x_{13} & 0 & 0 \\ 0 & 0 & x_{21} & x_{31} \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} -x_{13} \\ x_{12} \\ -x_{31} \\ x_{21} \end{pmatrix}.$$

These matrices are to be multiplied to column vectors from the left. The non-zero finely graded Betti numbers are

$$\beta_{0,(12,13,23,32)} = \beta_{0,(12,31,32)} = \beta_{0,(21,23,31,32)} = \beta_{0,(13,21,23)} = 1$$
$$\beta_{1,(12,13,23,31,32)} = \beta_{1,(12,21,23,31,32)} = \beta_{1,(13,21,23,31,32)} = \beta_{1,(12,13,21,23,32)} = 1$$
$$\beta_{2,(12,13,21,23,31,32)} = 1\,,$$

where, for example, $(12, 13, 23, 32)$ is the squarefree monomial $x_{12}x_{13}x_{23}x_{32}$ corresponding to the point $(0,0,0)$. Note that we are resolving the ideal $M(J)^*$ rather than the quotient $S/M(J)^*$ (which would yield a shift of $+1$ in the first coordinate of each Betti number). The non-zero coarsely graded Betti numbers are then

$$\beta_{0,3} = \beta_{0,4} = 2, \ \beta_{1,5} = 4, \ \beta_{2,6} = 1\,.$$

7. Further remarks and open questions

Having constructed cellular resolutions of ideals arising from regular mixed subdivisions of dilated simplices, a natural question to ask is if the assumption of regularity is really necessary. The relevant properties of our subdivisions were established by considering them as induced by arrangements of tropical hyperplanes, and hence these subdivisions were always regular. However, the construction of a labeled complex from an arbitrary mixed subdivision of $n\Delta_{d-1}$ still makes sense, and it is an open question (as far as we know) whether these also support cellular resolutions. As a special case, in light of Proposition 4.2 we can ask whether *any* fine mixed subdivision of $n\Delta_{d-1}$ supports a minimal cellular resolution of $\langle x_1, \ldots, x_d \rangle^n$.

A connection to tropical geometry is provided by the *tropical oriented matroids* of Ardila and Develin from [1]. There the authors introduce an axiomatic approach to the study of (fine) types, with a list of properties which they show are satisfied by the collection of fine types arising from an arrangement of tropical hyperplanes. It is conjectured that all abstract oriented matroids are realized by arbitrary subdivisions, and if this were the case we might think of the ideals described in the previous paragraph as 'tropical oriented matroid ideals'. A further task would be to relate the algebraic properties of these ideals with the combinatorial properties of the underlying matroid, in the spirit of [22].

As mentioned above, another unresolved question is to characterize which monomial ideals arise as coarse type ideals for some tropical hyperplane arrangement. We have seen that certain necessary properties are easy to deduce but it seems difficult to provide a complete classification. Do these ideals fit into some other well-known class?

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